STABLE INDECOMPOSABILITY OF THREE-MANIFOLDS

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ABSTRACT. We show that a variant of a recent result of Kwasik-Schultz [3] about stable indecomposability of three-manifolds is a direct consequence of results of Kotschick, Löh and Neofytidis [2, 1].

The following result was very recently proved by Kwasik and Schultz; see [3, Theorem A]:

Theorem 1. *Let M be a closed oriented 3-manifold that is not a Cartesian product. Then there is no closed oriented manifold N of dimension ≤ 3 such that $M \times N$ decomposes as a Cartesian product of surfaces and the circle.*

We show that the following variant of this theorem follows directly from [2, 1]:

Theorem 2. *Let M be a closed oriented 3-manifold that is not finitely covered by a Cartesian product. Then there is no closed oriented manifold N, of any dimension, such that $M \times N$ decomposes as a Cartesian product of surfaces and the circle.*

This is both weaker and stronger than Theorem 1. It is weaker in that we not only assume that M is not a product, but make the stronger assumption that M is not finitely covered by a product. It is stronger in that it does not require the assumption $\dim(N) \leq 3$.

1. PROOF OF THEOREM 2

Recall that a closed oriented 3-manifold is rationally essential if and only if it has an aspherical summand in its Kneser–Milnor decomposition; cf. [2, Theorem 3]. With the additional assumption of rational essentialness one has the following much stronger conclusion than in Theorems 1 and 2:

Proposition 3. *Let M be a rationally essential closed oriented 3-manifold that is not finitely covered by a Cartesian product. Then there is no closed oriented manifold N such that $M \times N$ admits a non-zero degree map from a Cartesian product of surfaces and the circle.*

Proof. Since M is rationally essential and not finitely covered by a product, it is not dominated by a product by [2, Proposition 1]. Therefore, the conclusion follows from [1, Theorem 2.3]. $\square$

Most cases of Theorem 2 follow from Proposition 3. The missing cases concern the rationally inessential manifolds that are not finitely covered by products. By [2, Theorem 3], a rationally inessential M is finitely covered by some $\#_k(S^1 \times S^2)$ for $k \geq 0$. Moreover, the case $k = 1$ is excluded by the assumption that M is not finitely covered by a product.

If $k \geq 2$, then $\pi_2(M) = H_2(\widetilde{M}; \mathbb{Z})$ is not finitely generated since $\widetilde{M}$ is the universal covering of $\#_k(S^1 \times S^2)$. For such M the conclusion of Theorem 2 follows from the fact that for a product

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of closed orientable surfaces and the circle, the second homotopy group is finitely generated, with generators the S^2-factors in the product.

Finally, if $k = 0$, then M is finitely covered by S^3 and the universal covering of any $M \times N$ splits as $S^3 \times \widetilde{N}$. In particular, $H_3(\widetilde{M \times N}; \mathbb{Z}) \neq 0$. For such an M the conclusion of Theorem 2 follows from the fact that for a product of closed orientable surfaces and the circle, the universal covering is a product of two-spheres and Euclidean spaces; in particular its third homology vanishes.

This completes the proof of Theorem 2.

References

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