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ON STABILITY OF NON-DOMINATION UNDER TAKING PRODUCTS

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Abstract. We show that non-domination results for targets that are not dominated by products are stable under Cartesian products.

1. Motivation

If M and N are closed oriented manifolds of the same dimension, we say that M *dominates* N, and we write $M \geq N$, if there is a continuous map $f\colon M \longrightarrow N$ of non-zero degree. The existence of such a dominant map is a property of the homotopy types of M and N, and it has been known since the pioneering work of Hopf [11] that for such a map f the pullback f^* is an injection of rational cohomology algebras, and that f_* is virtually surjective on the fundamental group. However, the existence of an injective algebra homomorphism $H^*(N;\mathbb{Q}) \longrightarrow H^*(M;\mathbb{Q})$ and of a virtually surjective homomorphism $\pi_1(M) \longrightarrow \pi_1(N)$ is usually far from sufficient for $M \geq N$.

Motivated by the work of Gromov [7,8] in particular, (non-)domination between manifolds has in recent years been studied in several different contexts, using a variety of techniques from topology, geometry, and group theory; see for example [4,5,7,8,12] and the references given there. An idea due to Thurston [16] and Gromov [7] is to study numerical invariants I of manifolds that are monotone under maps of non-zero degree, so that $M \geq N$ implies $I(M) \geq I(N)$. Then, whenever one can compute or estimate I and prove $I(M) < I(N)$ for some specific manifolds, one concludes that M does not dominate N. The simplest example of such an invariant is the cuplength in rational cohomology, which is monotone by the result of Hopf mentioned before. A more subtle monotone invariant – of geometric rather than algebraic origin – is the simplicial volume $\| \cdot \|$ defined by Gromov [7]. In general, monotone invariants are closely connected to functorial semi-norms on homology [6,8,15].

According to Gromov, the simplicial volume has a major deficiency: its lack of multiplicativity. In fact, he proved in [7] that the simplicial volume is approximately multiplicative for Cartesian products, and it is known that it is not strictly multiplicative [3]. However, approximate multiplicativity is not good enough to

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obtain stable non-domination results. Indeed, suppose that $0 < \|M\| < \|N\|$ for some specific M and N. Then $M \not\geq N$, but it is unclear whether the d-fold product $M^{\times d}$ may dominate $N^{\times d}$ for some $d \geq 2$, or not. The approximate multiplicativity does not rule out the possibility that, as a function of the number of factors, the simplicial volume of direct products of M might grow faster than that of direct products of N, so that the former eventually surpasses the latter.

Invariants that are strictly multiplicative – or strictly additive, like the cuplength – do not have this deficiency: if $I(M) < I(N)$, then $I(M^{\times d}) < I(N^{\times d})$, so that $M^{\times d} \not\geq N^{\times d}$ for all $d \geq 1$. In this case the non-domination result $M \not\geq N$ is stable under Cartesian products.

Gromov [8] suggested that many manifolds N might have the property that they cannot be dominated by a non-trivial product $M = M_1 \times M_2$. This conjecture has since been verified [12], and there are now lots of examples of manifolds that are known not to be dominated by products [12–14, 17]. We will see here that in general non-domination results for targets that cannot be dominated by products are stable under Cartesian products. This is interesting in its own right, and also has geometric applications [17].

Conventions. Throughout this paper, the word manifold means a connected closed oriented non-empty topological manifold; we denote the rational fundamental class of a manifold M by $[M]$. A product of manifolds is always a non-trivial product, so no factor is a point.

2. Results

Our first result is that for targets that are not dominated by products, the loss of information in taking products discussed in the previous section does not occur.

Theorem 2.1. *Suppose M and N are n-manifolds, and that N is not dominated by a product. Then for any $d \geq 2$ we have $M^{\times d} \geq N^{\times d}$ if and only if $M \geq N$.*

In a similar spirit, taking Cartesian products with arbitrary manifolds preserves non-domination for targets that are not dominated by products.

Theorem 2.2. *Suppose M and N are n-manifolds, and that N is not dominated by a product. Then for any manifold W, we have $M \times W \geq N \times W$ if and only if $M \geq N$.*

Note that W may very well have trivial simplicial volume. Even if one deduces $M \not\geq N$ from $\|M\| < \|N\|$, this theorem shows that multiplying with W preserves non-domination, while killing the simplicial volume if $\|W\| = 0$.

Finally, controlling the dimensions of the factors in a product, we have the following:

Theorem 2.3. *Let N be an n-manifold that is not dominated by a product. Then there is no manifold V for which the product $N \times V$ can be dominated by a product $P = X_1 \times \ldots \times X_s$ that satisfies $\dim X_j < n$ for all $j \in \{1, \ldots, s\}$.*

3. Proofs

The proofs of the above theorems all use the following lemma, which is a consequence of Thom's work [18] on the Steenrod problem.

Lemma 3.1. *Let N be an n-manifold that is not dominated by a product. If*
$$f\colon M_1 \times M_2 \longrightarrow N$$
is a continuous map, then for all $i \in \{1, \dots, n-1\}$ the map
$$f_*\colon H_i(M_1;\mathbb{Q}) \otimes H_{n-i}(M_2;\mathbb{Q}) \longrightarrow H_n(N;\mathbb{Q})$$
induced by the homological cross-product and f is the zero map.

Proof. Because elements of $H_i(M_1;\mathbb{Q}) \otimes H_{n-i}(M_2;\mathbb{Q})$ are finite linear combinations of decomposable elements, and f_* is linear, it suffices to show $f_*(\alpha \otimes \beta) = 0$ for all $\alpha \in H_i(M_1;\mathbb{Q})$ and all $\beta \in H_{n-i}(M_2;\mathbb{Q})$. Again by the linearity of f_*, there is no loss of generality in replacing α and β by non-zero multiples. Thus we may assume that these are integral homology classes. By Thom's result [18], after replacing the integral classes α and β by suitable non-zero multiples, there are continuous maps $g_j\colon X_j \longrightarrow M_j$ defined on manifolds X_j of dimensions i and $n-i$ respectively, such that $(g_1)_*[X_1] = \alpha$ and $(g_2)_*[X_2] = \beta$. It follows that
$$f_*(\alpha \otimes \beta) = \big(f \circ (g_1 \times g_2)\big)_*[X_1 \times X_2] \ .$$
This must vanish, because otherwise the map $f \circ (g_1 \times g_2)\colon X_1 \times X_2 \longrightarrow N$ would have non-zero degree, contradicting the assumption on N. ☐

Using Lemma 3.1, we now prove the theorems stated in the previous section.

Proof of Theorem 2.1. If $M \geq N$, then clearly $M^{\times d} \geq N^{\times d}$ for all $d \geq 2$. Conversely, suppose that $g\colon M^{\times d} \longrightarrow N^{\times d}$ has non-zero degree for some $d \geq 2$. We consider the composition $f = p_1 \circ g$, where p_1 is the projection to the first factor. Then f_* is surjective in rational homology. Since we assumed that N is not dominated by a product, Lemma 3.1 tells us that, in degree n, the map f_* vanishes on tensor products of homology vector spaces of non-zero degree. It follows that for at least one of the inclusions $i\colon M \longrightarrow M^{\times d}$ of a factor of $M^{\times d}$, the composition $f \circ i$ has non-zero degree, and thus $M \geq N$. ☐

Proof of Theorem 2.2. If $M \geq N$, then clearly $M \times W \geq N \times W$ for all manifolds W. Conversely, suppose that $f\colon M \times W \longrightarrow N \times W$ has non-zero degree for some W. We consider the induced map f_* on $H_n(\,\cdot\,;\mathbb{Q})$ in terms of the Künneth decompositions of the domain and of the target:
$$f_*\colon H_n(M;\mathbb{Q}) \oplus \mathcal{M}_1 \oplus H_n(W;\mathbb{Q}) \longrightarrow H_n(N;\mathbb{Q}) \oplus \mathcal{M}_2 \oplus H_n(W;\mathbb{Q}) \ ,$$
where $\mathcal{M}_i$ denotes the direct sum of tensor products of homology vector spaces in non-zero degrees.

Since we assumed that N is not dominated by a product, Lemma 3.1 tells us that $f_*(\mathcal{M}_1)$ is contained in $\mathcal{M}_2 \oplus H_n(W;\mathbb{Q})$. If we assume for a contradiction that $M \ngeq N$, then the same is true for $f_*(H_n(M;\mathbb{Q}))$.

Because f_* is surjective, we conclude that there is an $\alpha_0 \in H_n(W;\mathbb{Q})$ such that $f_*(\alpha_0) = [N] \neq 0$ holds in the quotient vector space
$$Q = H_n(N \times W;\mathbb{Q}) / f_*(H_n(M;\mathbb{Q}) \oplus \mathcal{M}_1) \ .$$
Note that Q is of finite, non-zero, dimension.

Now we think of α_0 as being in the target of f_*. By surjectivity of f_*, the class α_0 is in its image, so there exists an $\alpha_1 \in H_n(W;\mathbb{Q})$ satisfying $f_*(\alpha_1) = \alpha_0$ in Q (though not necessarily in $H_n(N \times W;\mathbb{Q})$). We proceed inductively to find $\alpha_{i+1} \in H_n(W;\mathbb{Q})$ with the property that $f_*(\alpha_{i+1}) = \alpha_i$ in Q. The assumptions that N is not dominated by a product, or by M, imply at every step that α_i does not vanish in the quotient Q.

Since Q is finite-dimensional, there is a minimal $k \in \mathbb{N}$ such that $\alpha_0, \ldots, \alpha_k$ are linearly dependent in Q. There are then $\lambda_i \in \mathbb{Q}$ with $\lambda_k \neq 0$ such that

$$\lambda_k \alpha_k + \ldots + \lambda_0 \alpha_0 = 0 \in Q \ .$$

We now take the left-hand side of this equation, considered as an element of $H_n(W;\mathbb{Q}) \subset H_n(M \times W;\mathbb{Q})$, and apply f_* to it to obtain

$$\lambda_k \alpha_{k-1} + \ldots + \lambda_1 \alpha_0 + \lambda_0 [N] \in f_*(H_n(M;\mathbb{Q}) \oplus \mathcal{M}_1) \ .$$

If $\lambda_0 = 0$, then this contradicts the minimality of k. If $\lambda_0 \neq 0$, then we reach the conclusion that in $H_n(N \times W;\mathbb{Q})$ the generator $[N] \in H_n(N;\mathbb{Q})$ is a linear combination of $\lambda_k \alpha_{k-1} + \ldots + \lambda_1 \alpha_0 \in H_n(W;\mathbb{Q})$ and of elements in

$$f_*\big(H_n(M;\mathbb{Q}) \oplus \mathcal{M}_1\big) \subset \mathcal{M}_2 \oplus H_n(W;\mathbb{Q}) \ .$$

This contradicts the Künneth decomposition, and hence proves $M \geq N$. □

Proof of Theorem 2.3. Suppose $g\colon X_1 \times \ldots \times X_s \longrightarrow N \times V$ is a continuous map, and consider the composition $f = p_1 \circ g$. The assumptions that N is not dominated by a product and that $\dim X_j < n$ for all j imply, as in the proof of Lemma 3.1, that f_* is the zero map in degree n. Therefore, g has degree zero. □

4. Discussion

4.1. **Applications of the cuplength.** It is not clear to what extent the assumption that N is not dominated by a product is necessary in the above theorems. While it is crucial for our proofs, this could be an artifact of our method. Indeed, there are cases of targets N which are dominated by products, and still one can prove our results for them. We now do this for tori, using the cuplength.

Recall that the cuplength of M, denoted $\mathrm{cl}(M)$, is the maximal number k for which there are classes $\alpha_1, \ldots, \alpha_k \in H^*(M;\mathbb{Q})$ of positive degrees with the property that $\alpha_1 \cup \ldots \cup \alpha_k \neq 0 \in H^*(M;\mathbb{Q})$. This is monotone under maps of non-zero degree by [11].[1] The compatibility of the Künneth decomposition with the cup product implies

$$\mathrm{cl}(M \times W) = \mathrm{cl}(M) + \mathrm{cl}(W) \ . \tag{1}$$

The following is easy and well known.

Lemma 4.1. *An n-manifold M dominates T^n if and only if there is an injective algebra homomorphism $H^*(T^n;\mathbb{Q}) \longrightarrow H^*(M;\mathbb{Q})$, equivalently, if $\mathrm{cl}(M) = n$.*

So this is a case where the algebraic necessary condition for domination derived from rational cohomology is also sufficient.

Lemma 4.1 combined with (1) tells us that Theorem 2.2 holds for $N = T^n$. Furthermore, we have:

Proposition 4.2. *If M_1 and M_2 are manifolds of dimensions m_1 and m_2 respectively, then $M_1 \times M_2 \geq T^{m_1+m_2}$ if and only if $M_1 \geq T^{m_1}$ and $M_2 \geq T^{m_2}$.*

In particular, Theorem 2.1 also holds for $N = T^n$.

[1] Hopf did not use cohomology, but formulated the conclusion in terms of the *Umkehr* map on intersection rings.

4.2. **Infinite products.** Gromov has suggested that some non-domination results should extend to infinite products, following his perspective on infinite products and related topics [1, 2, 9][10, Section 5].

By increasing the number d of factors in $P^{\times d}$, one would naively end up with a countably infinite product $P^{\times\infty}$, without any extra structure. A better way of looking at infinite products is probably to pick a (discrete, countable) group Γ, and to look at the space $P^\Gamma = \mathrm{Map}(\Gamma, P)$, equipped with the natural shift action of Γ. Now in formulating what $P^\Gamma \not\geq N^\Gamma$ might mean, one should only consider Γ-equivariant continuous maps between these product spaces.

The main issue is of course that for maps between these infinite-dimensional manifolds there is no naive, geometric, notion of degree. Instead, one should make full use of equivariance and define domination via surjectivity in a suitable homology theory, perhaps without necessarily attempting to define a degree.

References

[1] Mélanie Bertelson, *Topological invariants for discrete group actions*, Lett. Math. Phys. **69** (2004), 147–156, DOI 10.1007/s11005-004-9188-x. MR2104441 (2005m:37057)

[2] Mélanie Bertelson and Misha Gromov, *Dynamical Morse entropy*, Modern dynamical systems and applications, Cambridge Univ. Press, Cambridge, 2004, pp. 27–44. MR2090764 (2005m:37056)

[3] Michelle Bucher-Karlsson, *The simplicial volume of closed manifolds covered by* $\mathbb{H}^2 \times \mathbb{H}^2$, J. Topol. **1** (2008), no. 3, 584–602, DOI 10.1112/jtopol/jtn012. MR2417444 (2009i:53025)

[4] James A. Carlson and Domingo Toledo, *Harmonic mappings of Kähler manifolds to locally symmetric spaces*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 173–201. MR1019964 (91c:58032)

[5] Christopher Connell and Benson Farb, *The degree theorem in higher rank*, J. Differential Geom. **65** (2003), no. 1, 19–59. MR2057530 (2005d:53064)

[6] Diarmuid Crowley and Clara Löh, *Functorial seminorms on singular homology and (in)flexible manifolds*, Algebr. Geom. Topol. **15** (2015), no. 3, 1453–1499, DOI 10.2140/agt.2015.15.1453. MR3361142

[7] Michael Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 5–99 (1983). MR686042 (84h:53053)

[8] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)]; With appendices by M. Katz, P. Pansu and S. Semmes; Translated from the French by Sean Michael Bates. MR1699320 (2000d:53065)

[9] Misha Gromov, *Topological invariants of dynamical systems and spaces of holomorphic maps. I*, Math. Phys. Anal. Geom. **2** (1999), no. 4, 323–415, DOI 10.1023/A:1009841100168. MR1742309 (2001j:37037)

[10] Misha Gromov, *Spaces and questions*, Geom. Funct. Anal. **Special Volume** (2000), 118–161. GAFA 2000 (Tel Aviv, 1999). MR1826251 (2002e:53056)

[11] Heinz Hopf, *Zur Algebra der Abbildungen von Mannigfaltigkeiten* (German), J. Reine Angew. Math. **163** (1930), 71–88, DOI 10.1515/crll.1930.163.71. MR1581232

[12] D. Kotschick and C. Löh, *Fundamental classes not representable by products*, J. Lond. Math. Soc. (2) **79** (2009), no. 3, 545–561, DOI 10.1112/jlms/jdn089. MR2506686 (2011b:53089)

[13] D. Kotschick and C. Löh, *Groups not presentable by products*, Groups Geom. Dyn. **7** (2013), no. 1, 181–204, DOI 10.4171/GGD/180. MR3019079

[14] D. Kotschick and C. Neofytidis, *On three-manifolds dominated by circle bundles*, Math. Z. **274** (2013), no. 1-2, 21–32, DOI 10.1007/s00209-012-1055-3. MR3054316

[15] C. Löh, *Finite functorial semi-norms and representability*, Int. Math. Res. Notices (2015), doi: 10.1093/imrn/rnv250.

[16] J. Milnor and W. Thurston, *Characteristic numbers of* 3*-manifolds*, Enseignement Math. (2) **23** (1977), no. 3-4, 249–254. MR0500978 (58 #18462)

[17] C. Neofytidis, *Fundamental groups of aspherical manifolds and maps of non-zero degree*, preprint arxiv:1507.07778v1 [math.GT] 28Jul2015.

[18] René Thom, *Quelques propriétés globales des variétés différentiables* (French), Comment. Math. Helv. **28** (1954), 17–86. MR0061823 (15,890a)

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