Lebesgue's Differentiation Theorem via Maximal Functions

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Recall the classical FTC: if (a, b) is an interval in $\mathbb{R}$, $f\colon (a, b) \to \mathbb{R}$ is **continuous** and, for $x \in (a, b)$,

$$F(x) := \int_a^x f(y)\,\mathrm{d}y\,,$$

then for all $x \in (a, b)$, F is differentiable at x and

$$F'(x) = f(x)\,.$$

Now we might consider:

- How can we phrase this for open sets $\Omega \subset \mathbb{R}^n$, $n > 1$?
- What if $f \in L^1(a, b)$, $f \in L^1(\Omega)$?

Differentiation of general integrals

The derivative of an integral is the "limit of averages"

Note that for $f\colon (a,b) \to \mathbb{R}$ continuous and F defined as before, for $x \in (a,b)$ we have

$$F'(x) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(y)\,\mathrm{d}y = f(x)\,.$$

- Now let Ω be a general open subset in $\mathbb{R}^n$ (or $\Omega = \mathbb{R}^n$).
- It therefore makes sense to consider, for $x \in \Omega$, the quantity

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,\mathrm{d}y \quad \text{where } B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}\,.$$

- Interested in limit of this quantity as $r \to 0$.
- It is easy to show that if f is continuous, then limit is $f(x)$.

Theorem (Lebesgue's Differentiation Theorem)

Let $f \in L^1(\Omega)$. Then for almost all $x \in \Omega$, we have

$$\lim_{r \searrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,\mathrm{d}y = f(x)\,.$$

The Hardy-Littlewood Maximal Function

- Focus on $\Omega = \mathbb{R}^n$ (otherwise just let $f = 0$ outside Ω).
- To prove the Theorem, we need to get estimates on integral averages of balls.
- Hence, for $f \in L^1(\mathbb{R}^n)$, define the **Maximal Function** *Mf* of *f* as

$$(Mf)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, \mathrm{d}y \, .$$

Theorem (A “weak-type” inequality)

Let $f \in L^1(\mathbb{R}^n)$. For any $t > 0$ we have

$$\left|\{x \in \mathbb{R}^n : (Mf)(x) > t\}\right| \leq \frac{3^n}{t} \int_{\mathbb{R}^n} |f(y)| \, \mathrm{d}y \, .$$

Proof of the estimate

A covering lemma

Lemma (Vitali)

*Let $E \subset \mathbb{R}^n$ be the union of a **finite** number of balls $B(x_i, r_i)$, $i = 1, 2 \dots k$. Then there exists a subset $I \subset \{1, \dots k\}$ such that the balls $B(x_i, r_i)$ with $i \in I$ are **pairwise disjoint**, and*

$$E \subset \bigcup_{i \in I} B(x_i, 3r_i).$$

- Let $A_t := \{x \in \mathbb{R}^n : (Mf)(x) > t\}$. Can show this is a Borel set.
- By definition of Mf, for every $x \in A_t$ there is a ball $B(x, r_x)$ with

$$\frac{1}{|B(x, r_x)|} \int_{B(x,r_x)} |f(y)| \, \mathrm{d}y > t \qquad \left(\Rightarrow |B(x, r_x)| < t^{-1} \int_{B(x,r_x)} |f| \right)$$

- Let $K \subset A_t$ be compact.
- Then $\{B(x, r_x)\}_{x \in A_t}$ is a cover of K. So there exists a finite subcover.

Proof of the estimate

- By the Covering Lemma, there is a **disjoint** finite subfamily $\{B(x_i, r_i)\}_{i=1}^k$ such that

$$K \subset \bigcup_{i=1}^{k} B(x_i, 3r_i)\,.$$

- Hence we have

$$|K| \leq 3^n \sum_{i=1}^{k} |B(x_i, r_i)| \leq \frac{3^n}{t} \sum_{i=1}^{k} \int_{B(x_i, r_i)} |f(y)|\,\mathrm{d}y \leq \frac{3^n}{t} \int_{\mathbb{R}^n} |f(y)|\,\mathrm{d}y\,.$$

- Lebesgue Measure is "inner regular" i.e. for any Borel set E,

$$|E| = \sup\{\,|K| : K \subset E \text{ and } K \text{ compact}\}\,.$$

- Hence get upper bound for A_t.

Proof of Lebesgue's Differentiation Theorem

- We want to show for almost all $x \in \mathbb{R}^n$,

$$\limsup_{r \searrow 0} ⨍_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y = 0 \,.$$

- Take a continuous function $g \in L^1(\Omega)$. Then add and subtract $g(y) - g(x)$ and use the triangle inequality to get

$$\begin{aligned}
\overline{\sup_{r \searrow 0}} ⨍_{B(x,r)} |f - f(x)| &\leq \overline{\sup_{r \searrow 0}} ⨍_{B(x,r)} |g - g(x)| + \overline{\sup_{r \searrow 0}} ⨍_{B(x,r)} |(f - g) - (f(x) - g(x))| \\
&\leq \sup_{r > 0} ⨍_{B(x,r)} |f(y) - g(y)| - |f(x) - g(x)| \, \mathrm{d}y \\
&\leq M(|f - g|)(x) + |f(x) - g(x)|
\end{aligned}$$

- Now fix $\epsilon > 0$. Then

$$\limsup_{r \searrow 0} ⨍_{B(x,r)} |f - f(x)| > \epsilon \Rightarrow \; M(|f - g|)(x) > \tfrac{\epsilon}{2} \text{ OR } |f(x) - g(x)| > \tfrac{\epsilon}{2} \,.$$

Proof of Lebesgue's Differentiation Theorem

$$\left|\{x : |f(x) - g(x)| > \tfrac{\epsilon}{2}\}\right| \leq \frac{2}{\epsilon} \int_{\mathbb{R}^n} |f(x) - g(x)| \, \mathrm{d}x \quad \text{(Tshebyshev)}$$

$$\left|\{x : M(|f - g|)(x) > \tfrac{\epsilon}{2}\}\right| \leq \frac{2 \cdot 3^n}{\epsilon} \int_{\mathbb{R}^n} |f(x) - g(x)| \, \mathrm{d}x \quad \text{(Theorem)}$$

So

$$\left|\left\{x : \limsup_{r \searrow 0} ⨍_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y > \epsilon\right\}\right| \leq \frac{C}{\epsilon} \int_{\mathbb{R}^n} |f - g|$$

- This holds for all continuous $g \in L^1(\mathbb{R}^n)$. But these are dense in $L^1(\mathbb{R}^n)$, so, for fixed ϵ, can make RHS arbitrarily small.
- So for all $m \in \mathbb{N}$, taking $\epsilon = 1/m$,

$$\left|\left\{x : \limsup_{r \searrow 0} ⨍_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y > 1/m\right\}\right| = 0$$

Proof of Lebesgue's Differentiation Theorem

- Call such sets E_m. Then $|E_m| = 0\ \forall m$.
- Now note
$$\left|\left\{x : \limsup_{r\searrow 0} \fint_{B(x,r)} |f(y) - f(x)|\,\mathrm{d}y > 0\right\}\right| = \left|\bigcup_{m\in\mathbb{N}} E_m\right| = 0\,.$$
- Hence
$$\limsup_{r\searrow 0} \fint_{B(x,r)} |f(y) - f(x)|\,\mathrm{d}y = 0\,.$$
  for almost all x.

Generalisations of this Theorem

- Result (and proof) also holds if we replace Lebesgue measure with any locally finite Borel ("Radon") measure μ on $\mathbb{R}^n$.
- If μ, ν Radon measures on $\mathbb{R}^n$, we can consider the **derivative of ν with respect to μ** as the function

$$\frac{d\nu}{d\mu}(x) := \lim_{r \searrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} .$$

Theorem (Besicovich Differentiation Theorem)

$\frac{d\nu}{d\mu}(x)$ exists in $[0, \infty]$ μ and ν almost everywhere. If we let

$$S := \left\{ x \in \mathbb{R}^n : \frac{d\nu}{d\mu}(x) = \infty \right\},$$

then $\mu(S) = 0$ and for all Borel sets E,

$$\nu(E) = \int_E \frac{d\nu}{d\mu}(x) \, d\mu(x) + \nu(E \cap S) .$$

Concluding remarks

- Maximal Functions have very useful applications in many branches of Mathematics (e.g. PDE Theory, Calculus of Variations, Harmonic Analysis...)
- There are many more interesting things that can be said about them. e.g. if $f \in L^p$ for $1 < p \leq \infty$, then $Mf \in L^p$ too.
- Moreover, it is also often useful to be able to exploit **pointwise** properties of integrable functions.
- Lebesgue's Differentiation Theorem is a powerful result in this context.

End of presentation