Bochner Spaces

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Motivation

We consider a parabolic system of partial differential equations

$$\begin{aligned} \partial_t u - \operatorname{div} \mathrm{DF}(\nabla u) &= 0 && \text{on} \quad (0, T) \times B = Q_T \\ u(t, \cdot) &= 0 && \text{on} \quad [0, T] \times \partial B \\ u(0, \cdot) &= u_0(\cdot) && \text{a. e.,} \end{aligned}$$

and postulate

$$\begin{aligned} u &: Q_T \to \mathbb{R}^{\mathrm{N}}, \\ F &: \mathbb{R}^{\mathrm{dN}} \to \mathbb{R}, \\ u_0 &\in W^{1,2}(B). \end{aligned}$$

What is the appropriate solutions space?

The Bochner Integral I

Let $(X, \|\cdot\|_X)$ be a Banach space.
A stair function is a $v : [0, T] \to X$,

$$v(t) = \sum_{k=1}^{N} \chi_{A_k}(t) x_k,$$

$$\text{for} \quad A_k : A_i \cap A_j = \emptyset \quad \text{and} \quad \bigcup_k A_k = [0, T].$$

If there exists a sequence $(u_n)_n$ of stair functions with

$$u_n(t) \to u(t), \quad \text{in} \quad X,$$

$$\text{and} \int_0^T \|u_n(t) - u(t)\|_X \,\mathrm{d}t \to 0, \quad \text{for} \quad n \to \infty,$$

it is Bochner measurable.

The Bochner Integral II

Now we can define

$$\int_0^T u \, \mathrm{d}t = \lim_{n\to\infty} \int_0^T u_n(t) \, \mathrm{d}t = \lim_{n\to\infty} \mathcal{L}^1(A_k^n) x_k^n.$$

And thus the space $L^p(0, T; X)$ of all functions u with

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X \, \mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

The space $C^m([0,T],X)$

Let u be m-times differentiable,

$$\|u\|_{C^m(X)} = \sum_{i=0}^{m} \max_{0 \le t \le T} |u^{(i)}(t)|,$$

and denote their space $C^m([0,T],X)$.

This space is Banach.

Properties

Let X, Y be Banach spaces, $1 \leq p < \infty$ and $m \in \mathbb{N}$. Then

a) $C^m([0,T],X)$ is dense in $L^p(0,T;X)$ and the embedding

$$C^m([0,T],X) \hookrightarrow L^p(0,T;X),$$

is continuous.

b) the set of all polynomials $v : [0,T] \to X$, i.e.

$$v(t) = \sum_{i=1}^{n} a_i t^i,$$

with $a_i \in X$ and $n \in \mathbb{N}$ is dense in both $C([0,T],X)$ and $L^p(0,T;X)$.

Properties cont.

(c) if the embedding $X \to Y$ is continuous, the embedding

$$L^r(0, T; X) \hookrightarrow L^q(0, T; Y), \quad \text{for} \quad 1 \leq q \leq r \leq \infty,$$

is also continuous.

(d) The closure

$$\overline{C_0^\infty([0, T], X)}^{\|\cdot\|_{L^p(X)}}$$

can be identified by $L^p(0, T; X)$.

Negative Sobolev Spaces $W^{-k,q}$

Let $m \in \mathbb{N}$, $1 < p < \infty$ and p' its Hölder conjugate.

$$\|v\|_{-m,p} = \sup_{\substack{u \in W_0^{m,p}(\Omega), \\ \|u\|_{m,p} \leq 1}} \left| \int_Q u(x)v(x)\,\mathrm{d}x \right|.$$

Now we denote $W^{-m,p}(\Omega)$ as the closure of $L^{p'}(\Omega)$, i.e.

$$\overline{L^{p'}(\Omega)}^{\|\cdot\|_{-m,p}} \equiv W_0^{m,p}(\Omega)^\star \equiv W^{-m,p'}(\Omega).$$

Generalised derivative

Let X be Banach spaces, $u, w \in L^1(0,T;X)$. If w satisfies

$$\int_0^T \varphi^{(n)}(t)u(t)\,\mathrm{d}t = (-1)^n \int_0^T \varphi(t)w(t), \quad \text{for all} \quad \varphi \in C_0^\infty([0,T],\mathbb{R}),$$

then it is the n-th generalised derivative of u on $(0,T)$.

We denote $u^{(n)} = w$.

Dual Space of $L^p(0, T; X)$

Let X be a reflexive Banach space and $\frac{1}{p} + \frac{1}{p'} = 1$.
If $1 < p < \infty$, then $L^p(0, T; X)$ is reflexive and we have

$$L^p(0, T; X)^\star \cong L^{p'}(0, T; X^\star),$$

i.e. its dual space is isometrically isomorphic to the Hölder conjugate Bochner space of the dual $X^\star$.

Construction of the Solution Space

If we have $u \in L^p(0, T; X)$ with the generalised derivative

$$\partial_t u \in L^{p'}(0, T; X^\star)$$

form a real Banach space with the norm

$$\|u\|_{W^{p,q}} = \|u\|_{L^p(0,T;X)} + \|\partial_t u\|_{L^q(0,T;Z)}.$$

We denote this space $W^{p,q}(0, T; X, Z)$.

Application

We set $X = W_0^{1,p}(B)$ and $Z = W^{-1,p}(B)$.

Then we get

$$W^{p,q}(0,1;W_0^{1,p},W^{-1,p}) =$$

$$\left\{u \in L^p(0,T;W_0^{1,p}(B)) \;\middle|\; \partial_t u \in L^q(0,1;W^{-1,p}(B))\right\}.$$