Pointwise properties of Sobolev Functions

Affine Behaviour in an Integral Sense

Sebastian Engel

LMU München

München am 26.06.2014

Introduction

Consider the differential quotient of $f \in C^1(\Omega, \mathbb{R}^N)$ with $\Omega \subset \mathbb{R}^n$ open and bdd., let $x_0, y \in \Omega$ and $[x_0, y] \subset \Omega$:

$$\frac{u(y) - u(x_0)}{|y - x_0|}$$

(Note: differential quotient is everywhere pointwise-defined).
Aim:

- Weaken the differential quotient definition in an integral sense
- The differential quotient for Sobolev functions can be approximated ($\mathbb{L}^p$-limes) by the weak Jacobi matrix
- Using a Poincaré inequality and Lebesgue differentiation theorem (Application)

Recall:

- $W^{1,p}(\Omega, \mathbb{R}^N) := \left\{ u \in L^p(\Omega)^N \,\middle|\, \begin{array}{c} u_i \in L^p(\Omega) \\ u_i \text{ weak diff.} \\ i = 1, ..., N \end{array} \right\}$
- Weak Jacobi: Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, then there exist for a.a. $x_0 \in \Omega$ the weak derivative of u_i in x_0 in direction $e_j \in \mathbb{R}^n$ ($i = 1, ..., N$ and $j = 1, ..., n$), i.e. $D^j u_i(x_0)$. So we have the weak Jacobi $[\nabla u(x_0)]$ in the following way defined: $\left(D^j u_i(x_0) \right)_{\substack{i=1,\dots,N \\ j=1,\dots,n}} \in \mathbb{R}^{Nxn}$

Lebesgue Differentiation Theorem

Let $f \in \mathbb{L}^p(\Omega, \mathbb{R}^N)$ with $\Omega \subset \mathbb{R}^n$ open and $1 \leq p < \infty$, then for a.e. $x_0 \in \Omega$ (Lebesgue points), we have:

$$\lim_{r \to 0} \fint_{B(x_0,r)} \Big| f(y) - f(x_0) \Big|^p dy = 0$$

We even have:

$$\lim_{r \to 0} \fint_{B(x_0,r)} \Big| f(y) - \Big(\fint_{B(x_0,r)} f(z) dz \Big) \Big|^p dy = 0$$

Instead of balls, we can use cubes
(i.e. $Q(x_0, r) = ([x_0]_1 - \frac{r}{2}, [x_0]_1 + \frac{r}{2}) x ... x ([x_0]_n - \frac{r}{2}, [x_0]_n + \frac{r}{2}))$

Proof is using maximal functions (see Huettenseminar WS13, Dr. Soneji).

Poincaré Inequalities

Let $\Omega \subset \mathbb{R}^n$ bdd., connected and open with Lip.-boundary, $1 \leq p < \infty$.
Then there exists a const. $c = c(\Omega, p, N) > 0$ s.t. $\forall u \in W^{1,p}(\Omega, \mathbb{R}^N)$:

$$\int_\Omega \left| u(x) - \left(⨍_\Omega u \right) \right|^p dx \leq c \int_\Omega |\nabla u|^p dx$$

Proof uses

- An indirect argument (proof by contradiction) and
- The Rellich-Kondrachov Compactness Theorem for a specially constructed sequence of functions (see Evans,PDE page 290).

(Rellich-Kondrachov: $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$, $1 \leq q < p^* = \frac{pn}{n-p}$)

Poincaré Inequalities

Consider $1 \leq p < \infty$, $\Omega = Q(x_0, r)$ and $Q = Q(0,1)$ as unit cube.
By Poincaré $\exists c = c(Q, p, N) > 0$ s.t. $\forall u \in W^{1,p}(Q(x_0, r), \mathbb{R}^N)$:

$$\int_{Q(x_0,r)} \left| u(x) - \left(\fint_{Q(x_0,r)} u \right) \right|^p \leq cr^p \int_{Q(x_0,r)} |\nabla u|^p dx$$

(Proof in Ziemer, Weakly Differenitable Functions (page 126, theorem 3.4.1))

Pointwise Properties of Sobolev Functions

Affine behaviour in an integral sense

Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ open.
Then for $\mathfrak{L}^n - a.e.\ x_0 \in \Omega$:

$$\lim_{r \to 0} \fint_{Q(x_0,r)} \left| \frac{u(y) - u(x_0)}{r} - \frac{[\nabla u(x_0)](y - x_0)}{r} \right|^p dy = 0$$

with r small enough (i.e. $Q(x, r) \subset \Omega$; possible by openess).

Note: the affine map (weak Jacobi) $\left\{ \begin{matrix} [\nabla u(x_0)] : Q(x_0, r) \longrightarrow \mathbb{R}^N \\ y \mapsto \nabla u(x_0) y \end{matrix} \right\}$ exists

for $\mathfrak{L}^n - a.a.\ x_0 \in \Omega$

Proof:

Let $x_0 \in \Omega$ be an Lebesgue point for ∇u and u, i.e.

$$x_0 \in \bigcap_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq 1 \\ j \in \{1, \dots N\}}} \underbrace{\{\text{Leb. pts. of } D^\alpha u_j\}}_{\text{are } \mathfrak{L}^n - a.a.\ x \in \Omega}$$

Claim: For the next almost everywhere defined function on Ω, holds:

$$\Omega \ni x_0 \mapsto \fint_{Q(x_0,r)} [\nabla u(x_0)] y dy \overset{(!)}{=} [\nabla u(x_0)] x_0 \in \mathbb{R}^N$$

Claim Proof:

$$\fint_{Q(x_0,r)} [\nabla u(x_0)] y dy = \fint_{Q(x_0,r)} \left(\sum_{j=1,\dots,n} D^{\alpha_j} u_i(x_0) y_j \right)_{i=1,\dots,N} dy =$$

$$\left(\sum_{j=1,\dots,n} D^{\alpha_j} u_i(x_0) \fint_{Q(x_0,r)} y_j \right)_{i=1,\dots,N} dy =$$

$$[\nabla u(x_0)] \underbrace{\fint_{Q(x_0,r)} y dy}_{= \fint_{Q(0,r)} y + x_0 dy = 0 + x_0, \text{ see } (**)} = [\nabla u(x_0)] x_0$$

$$(**) : \left(\fint_{Q(0,r)} y_i dy \right)_i \underbrace{=}_{Fubini} \left(\frac{1}{Vol(Q(0,r))} \int_{-\frac{r}{2}}^{+\frac{r}{2}} dy_1 \ldots \underbrace{\int_{-\frac{r}{2}}^{+\frac{r}{2}} y_i dy_i}_{=0} \ldots \int_{-\frac{r}{2}}^{+\frac{r}{2}} dy_n \right)_i = 0$$

By Poincaré's inequality and claim (i.e. case $\Omega = Q(x_0, r)$) used for $\varphi : y \mapsto u(y) - [\nabla u(x_0)]y$, we get:

$$\fint_{Q(x_0,r)} \left| u(y) - \left(\fint_{Q(x_0,r)} u \right) - [\nabla u(x_0)](y - x_0) \right|^p dy =$$

$$\fint_{Q(x_0,r)} \left| \varphi(y) - \left(\fint_{Q(x_0,r)} \varphi \right) \right|^p dy =$$

$$= \fint_{Q(x_0,r)} \left|\varphi(y) - \left(\fint_{Q(x_0,r)} \varphi \right)\right|^p dy \underbrace{\leq}_{\text{Poincaré claim}}$$

$$cr^p \fint_{Q(x_0,r)} |\nabla\varphi|^p dx = cr^p \fint_{Q(x_0,r)} |\nabla u(y) - \nabla u(x_0)|^p dx$$

By the Lebesgue differentiation Theorem ($\nabla\varphi(\bullet) \in L^p(\Omega, \mathbb{R}^{Nxn})$) we get the conv. to zero!

Finally:

- Dividing both sides with r^p and using the fact of Lebesgue points,

we get for a.a. $x_0 \in \Omega$:

$$\lim_{r\to\infty} \fint_{Q(x_0,r)} \left| \frac{u(y) - (\fint_{Q(x_0,r)} u) - [\nabla u(x_0)](y - x_0)}{r} \right|^p dy = 0$$

- With W. Ziemer, Weakly Differenitable Functions (page 129, Theorem 3.4.2.) it is possible to replace $\fint_{Q(x_0,r)} u(y)dy$ with $u(x_0)$.
  - Ziemer proves our theorem even for weak Taylor series.
  - (Proof uses Lebesgue Differentiation Theorem, mollifiers and Lebesgue Dominated Convergence)

So we have for a.a. $x_0 \in \Omega$:

$$\lim_{r \to 0} \fint_{Q(x_0,r)} \left| \frac{u(y) - u(x_0) - [\nabla u(x_0)](y - x_0)}{r} \right|^p dy = 0$$

Conclusion:
this Theorem tells us: The Differential Quotient $\frac{u(y)-u(x_0)}{|y-x_0|}$ behaves like the affine map $y \mapsto \nabla u(x_0)y$ close to x_0 in an integral sense.

Morrey's Theorem

Let $f : \mathbb{R}^{Nxn} \to \mathbb{R}$ be a quasiconvex* function with $0 \leq f(A) < C(1 + |A|^p)$ for some $p \in [1, \infty)$. Then

$$F(u, \Omega) = \int_\Omega f(\nabla u) dx$$

is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega, \mathbb{R}^N)$.
Proof Idea:

- Prove this, first using $\Omega = Q$, $u_j \rightharpoonup Ax$ (affine map A).
- Use this property to obtain result in general case (by blowup methode).

Such affine map approximation like in Morrey's theorem is often used in Calculus of Variations.

*: $\int_Q f(A + \nabla\varphi) dx \geq f(A)$ $\forall A \in \mathbb{R}^{Nxn}$, $\forall \varphi \in W_{\mathbf{0}}^{\mathbf{1},\infty}(Q, \mathbb{R}^N)$