On Prandtl's 1945 model of turbulence: existence of weak solutions to the equations of unsteady turbulent pipe-flow

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Abstract

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, let $0 < T < +\infty$; define $Q = \Omega \times]0,T[$. We consider the following system of PDEs:

$$\frac{\partial u}{\partial t} - \operatorname{div}((\nu + \ell\sqrt{k})\nabla u) = f \quad \text{in} \quad Q, \tag{1}$$

$$\frac{\partial k}{\partial t} - \operatorname{div}((\mu + \ell\sqrt{k})\nabla k) = \ell\sqrt{k}|\nabla u|^2 - \frac{k\sqrt{k}}{\ell} \quad \text{in} \quad Q \tag{2}$$

($\nu, \mu = \text{const} \geq 0$, f given, $\ell = \ell(x)$ mixing length, $\ell(x) > 0$ for all $x \in \Omega$, $\ell(x) \to 0$ as $x \to \partial\Omega$). As a model for the fully developed turbulence, L. Prandtl, in 1945, postulated this system with $\nu = \mu = 0$ for general flow profiles. System (1), (2) characterizes the unidirectional flow through a pipe with cross section Ω.

We complete (1), (2) by boundary conditions on $\partial\Omega \times]0,T[$ and initial conditions in $\Omega \times \{0\}$. We prove the existence of a weak solution (u,k) ($k \geq 0$ a.e. in Q) to this boundary-initial value problem such that

$$\exists Q^* \subset Q: \quad \operatorname{mes}(Q^*) > 0, \quad k > 0 \quad \text{a.e. in } Q^*.$$