Lipschitz Truncation

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Framework

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- (i)Revision: p-Laplacian problem & difficulties
- (ii)About some analytic notions
  - Maximal function, A-fatness
  - Sobolev, Lipschitz spaces
- (iii) Lipschitz truncation: CUTTING THE GRADIENTS
  - Solution to (i)

Consider the p-Laplacian problem:

$$-\text{div}(|\mathbf{Dv}|^{p-2}\mathbf{Dv}) = \mathbf{F} \text{ in } \Omega \subset \mathbb{R}^N$$

$$\mathbf{v} = 0 \text{ on } \partial\Omega$$

where Ω has Lipschitz boundary, $p > 1$, $\mathbf{Dv}$ (symmetrized gradient).
Remark: Key assumption: $p > 1 \longrightarrow$ Reflexivity of $W_0^{1,p}(\Omega)^N$! Let $\mathcal{X}$ a suitable class of functions/distributions and $\varphi \in \mathcal{X}$.

Let $\{\mathbf{v}^n\} \subset \mathcal{X}$ be a sequence s.t.

(i)

$$\int_\Omega |\mathbf{D}\mathbf{v}^n|^{p-2}\mathbf{D}\mathbf{v}^n \cdot \mathbf{D}\varphi \, dx = \langle \mathbf{F}^n, \varphi \rangle$$

(ii)

$$\sup_n \int_\Omega |\mathbf{D}\mathbf{v}^n|^p \, dx < \infty$$

$\longrightarrow$ $\mathbf{v}^n \rightharpoonup \mathbf{v}$ weakly in $W_0^{1,p}(\Omega)^N$

(iii)

$$\langle \mathbf{F}^n, \varphi \rangle \longrightarrow \langle \mathbf{F}, \varphi \rangle$$

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Definition

Assume in the above situation $\mathbf{v}$ is also a weak solution to the p-Laplacian system. Then this system is said to satisfy the *weak stability property*.

- How to reach weak stability?
- Strict monotonicity of $T(X) \equiv |X|^{p-2}X$ and

$$\limsup_{n\to\infty}(T(D\mathbf{v}^n) - T(D\mathbf{v})) \cdot D(v^n - v)\,dx = 0$$

  imply $D\mathbf{v}^n \to Dv$ a.e. in Ω.
- Vitali's convergence theorem $\Rightarrow$ pass to the limit in the nonlinear term

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The simple case

$F_n, F \in (W_0^{1,p}(\Omega)^d)^*$ such that $F_n \to F$ strongly. Take $\varphi \equiv \mathbf{v} - \mathbf{v}^n$ and obtain

$$\int_\Omega (T(D\mathbf{v}^n) - T(D\mathbf{v}))(D(\mathbf{v}^n - \mathbf{v}))dx = \langle F^n, \mathbf{v}^n - \mathbf{v} \rangle - \int_\Omega T(D\mathbf{v}) \cdot D(\mathbf{v}^n - \mathbf{v})\, dx$$

The difficult case

Assume $F^n = \text{div}(G^n)$ with $G^n \to G$ strongly in $L^1(\Omega)^{d\times d}$. $\mathbf{u}^n = \mathbf{v} - \mathbf{v}^n$

$\longrightarrow \langle \text{div}(G^n), \mathbf{u}^n \rangle$, $-\langle G, \nabla \mathbf{u}^n \rangle$ have no clear meaning.

IDEA: Replace $\mathbf{u}^n$ by its *Lipschitz truncation*. Then uniform smallness of the integrand on sets where the Lipschitz truncation is not equal to $\mathbf{u}^n$ lead to

$$\limsup_{n\to\infty} ((T(D\mathbf{v}^n) - T(D\mathbf{v})) \cdot D(v^n - v))^\theta \, dx = 0$$

for some $\theta \in (0, 1]$.

The maximal function and Lipschitz spaces

Let $1 < p < \infty$.

$$M\colon L^1(\Omega) \ni f \mapsto (Mf)(x) \equiv \sup_{r>0:\ B(x,r)\subset\Omega} \fint_{B(x,r)} f d\mathcal{L}^n$$

is called the maximal function of $f \in L^1(\Omega)$.

- Example: Assume $f\colon \mathbb{R}^N \to \mathbb{R}$ harmonic. Then $(Mf)(x) = f(x)$. This is, any harmonic function is a fixed point of M.
- Note: If $1 < p < \infty$, the Hardy-Littlewood operator is a continuous operator $M\colon L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ $\longrightarrow$ by Hardy-Littlewood-Inequality

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A_1-property

Let $\Omega \subset \mathbb{R}^N$ be bounded. It fulfills a $A_1 \geq 1$ property iff there exists $A_1 \geq 1$ such that for all $x \in \Omega$

$$|B_{2dist(x,\Omega^C)}(x)| \leq A_1 \cdot |B_{2dist(x,\Omega^C}(x) \cap \Omega^C|$$

holds true.

Lipschitz functions and extensions

- $W^{1,\infty}(\Omega) = \text{Lip}(\Omega)$

Theorem

(Lipschitz extension) Assume $\Omega \subset \mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}^M$ be Lipschitz. Then there exists a Lipschitz function $\overline{f} : \mathbb{R}^N \to \mathbb{R}^M$ such that

$$\overline{f}|_\Omega = f$$

$$Lip(\overline{f}) \leq \sqrt{M} Lip(f)$$

Theorem (Acerbi, Fusco 1988)

Let $\Omega \subset \mathbb{R}^N$ have the A_1-property, $A \geq 1$. Let $\mathbf{v} \in W_0^{1,1}(\Omega)^N$. Then for every $\theta, \lambda >$ exist truncations $\mathbf{v}_{\theta,\lambda} \in W_0^{1,\infty}(\Omega)^N$ such that

- $||\mathbf{v}_{\theta,\lambda}||_\infty \leq \theta$
- $||\nabla \mathbf{v}_{\theta,\lambda}||_\infty \leq c_1 A_1 \lambda$

where c_1 only depends on the dimension N. Moreover, up to a nullset it holds

$$\{\mathbf{v}_{\theta,\lambda} \neq \mathbf{v}\} \subset \Omega \cap (\{M\mathbf{v} > \theta\} \cup \{M(\mathbf{v}) > \lambda\})$$

Theorem (Diening, Malek, Steinhauer 2006)

Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain which has the A_1-property. Let $\{\mathbf{u}^n\} \subset W_0^{1,p}(\Omega)^d$ such that

$$\mathbf{u}^n \rightharpoonup 0 \, in \, W_0^{1,p}(\Omega)^d$$

Set

$$\mathfrak{K} \equiv \sup_{n \in \mathbb{N}} ||\mathbf{u}^n||_{W^{1,p}(\mathbb{R}^N)} < \infty \,\&\, \gamma \equiv ||\mathbf{u}^n||_{L^p(\mathbb{R}^N}$$

Let $\theta_n > 0$ such that $\theta_n \to 0$ as $n \to \infty$ and

$$\frac{\gamma_n}{\theta_n} \to 0, \; n \to \infty$$

Set $\mu_j = 2^{2^j}$.

Then: There exists a sequence $\{\lambda_n\} \subset \mathbb{R}$, $\lambda_{n,j} > 0$ such that $\mu_j \leq \lambda_{n,j} \leq \mu_{j+1}$ and a sequence $\{\mathbf{u}^{n,j}\} \subset W_0^{1,\infty}(\Omega)^d$ such that

$$||\mathbf{u}^{n,j}||_\infty \leq \theta_n \to 0$$

$$||\nabla \mathbf{u}^{n,j}||_\infty \leq c\lambda_{n,j} \leq c\mu_{j+1}$$

and up to some nullset

$$\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\} \subset \Omega \cap (\{M\mathbf{u}^n > \theta\} \cup \{M\nabla \mathbf{u}^{n,j} > 2\lambda_{n,j}\})$$

and for all $j \in \mathbb{N}$ as $n \to \infty$:

$$\mathbf{u}^{n,j} \to 0 \,\text{strongly in}\, L^s(\Omega)^d \;\forall s \in [1,\infty]$$

$$\mathbf{u}^{n,j} \rightharpoonup 0 \,\text{weakly in}\; W_0^{1,s}(\Omega)^d \;\forall s \in [1,\infty)$$

$$\nabla \mathbf{u}^{n,j} \rightharpoonup^* \;\text{*-weakly in}\; L^\infty(\Omega)^d$$

and

$$||\nabla \mathbf{u}^{n,j} \chi_{\mathbf{u}^{n,j} \neq \mathbf{u}^n}||_{L^p(\Omega)} \leq c\frac{\gamma_n}{\theta_n}\mu_{j+1} + \mathfrak{K}c2^{-j/p}$$

Solution to the problem

- $\{\mathbf{u}^n\}$ fulfills the assumptions of the Lipschitz truncation theorem
- The sequence $\{\mathbf{u}^{n,j}\} \subset W_0^{1,\infty}(\Omega)^d$ are admissible test functions.
- Thus,

$$\int_\Omega (T(D\mathbf{v}^n) - T(D\mathbf{v})) \cdot (Du^{n,j})\, dx =$$

$$-\int_\Omega ((G^n - G) + G + T(D\mathbf{v})) \cdot D\mathbf{u}^{n,j}\, dx$$

- But: $G^n \to G$ strongly in $L^1(\Omega)^{d\times d}$ and $Du^{n,j} \rightharpoonup^* 0$ in $L^\infty(\Omega)^d$