VARIABLE EXPONENT TRACE SPACES

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ABSTRACT. The trace space of $W^{1,p(\cdot)}(\mathbb{R}^n \times [0,\infty))$ consists of those functions on $\mathbb{R}^n$ that can be extended to functions of $W^{1,p(\cdot)}(\mathbb{R}^n \times [0,\infty))$ (as in the fixed-exponent case). Under the assumption that p is globally log-Hölder continuous, we show that the trace space depends only on the values of p on the boundary. In our main result we show how to define an intrinsic norm for the trace space in terms of a sharp-type operator.

1. Introduction

In this article we present a new approach to trace spaces. Our philosophy is to move away as little as possible from the definition of trace space as consisting of those functions which can be extended to the ambient space. The motivation for pursuing this line of investigation is that it provides us with more robust results and methods. We are especially interested in Sobolev spaces with variable exponent. What makes variable exponent spaces stand apart particularly in the current context is that they are not translation invariant, in contrast to their classical counter-parts. A glance at the classical approaches (due to Lions, Peetre and others, see, e.g., [3, Section 7], [4, Section 7] and references therein) shows that translation invariance is in many situations at the heart of the matter, starting with the idea that we can define a norm as a Bochner integral of a function from the real line to a Banach space. We believe that our approach can be used also when dealing with other non-translation invariant generalizations of Sobolev spaces.

On an intuitive level we get the variable exponent spaces by replacing the energy (modular)

$$\int_\Omega |f(x)|^p dx \quad \text{by} \quad \int_\Omega |f(x)|^{p(x)} dx,$$

where $p(x)$ is some function. Exact definitions are given below. Let us review some of the major reasons for why variable exponent spaces have attracted quite a bit of attention lately (see [12] for a bibliography of over a hundred titles on this topic from the last five years). Variable exponent spaces are connected to variational integrals with non-standard growth and coercivity conditions [2, 30]. These non-standard variational problems are related to modeling of so-called electrorheological fluids [1, 28] and also appear in a model for image restoration [5]. Another reason for the recent interest is that the "right" framework for variable exponent spaces was discovered: the log-Hölder continuity condition was found to be sufficient for many regularity properties of the spaces, starting with the local boundedness of the maximal operator [9].

Obviously, the study of trace spaces is of great importance for the theory of partial differential equations. Indeed, a partial differential equation is in many cases solvable if and only if the boundary values are in the corresponding trace space, see e.g. [15]. The first

LD supported in part by the Landesstiftung Baden-Württemberg, Project *Intelligent Materials*.

PH supported by the Research Council of Norway, Project 160192/V30.

appearance of trace spaces in the context of Sobolev spaces with variable exponents $W^{1,p(\cdot)}$ is in [13, 14], where the solvability of the Laplace equation $-\Delta u = f$ on the half-space with given boundary values is studied. The definition of trace spaces by Diening and Růžička [13, 14] matches ours in Section 3. However, they avoided studying trace spaces, considering them instead as abstract objects. To describe these spaces, especially by an intrinsic norm, is the purpose of this article.

We now get back to variable exponent trace spaces. Another more concrete form of the problems related to translation non-invariance can be found by looking at the well-known intrinsic characterization of the fixed-exponent trace space of $W^{1,p}(\mathbb{H})$, where $\mathbb{H}$ is the open half space $\mathbb{R}^n \times (0,\infty)$: f is in the trace space if and only if

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p-1}}\,dy\,dx < \infty.$$

We would like to have the exponent vary with the location in the space, but clearly p in the previous formula can be replaced by neither $p(x)$ nor $p(y)$. There are similar difficulties with generalizing the formulae of other fractional order spaces, such as Besov spaces or Nikol'skii spaces. In this article we present an alternative conceptualization of the trace space problem. We try to present our approach in as simple a form as possible, in order to convey the main ideas, and hopefully to allow others to adapt them to other settings.

The main results of this paper are summarized in the following theorem.

Theorem 1.1. *Let $p\colon \mathbb{R}^n \to [1,\infty)$ be a variable exponent with $1 < \inf p \leqslant \sup p < \infty$ which is globally $\log$-Hölder continuous, i.e. assume that there exist $c > 0$ and $p_\infty > 1$ such that*

$$|p(x)-p(y)| \leqslant \frac{c}{\log(1/|x-y|)} \quad \text{and} \quad |p(z)-p_\infty| \leqslant \frac{c}{\log(e+|z|)}$$

for all points $x,y,z \in \mathbb{R}^n$ with $|x-y| < \frac{1}{2}$. Let q be an extension of p to $\overline{\mathbb{H}}$ which is also globally $\log$-Hölder continuous. Then the function f belongs to the trace space $\operatorname{Tr} W^{1,q(\cdot)}(\mathbb{H})$ if and only if

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx + \int_0^1 \int_{\mathbb{R}^n} \Big(\tfrac{1}{r} M^\sharp_{B^n(x,r)} f\Big)^{p(x)} dx\,dr < \infty,$$

where $M^\sharp_{B^n(x,r)}$ denotes the sharp operator,

$$M^\sharp_{B^n(x,r)} f = \fint_{B^n(x,r)} \Big| f(y) - \fint_{B^n(x,r)} f(z)\,dz \Big|\,dy.$$

In particular, the trace space depends only on the value of the exponent on the boundary.

We prove this result in a piece-meal fashion. We start in Section 2 by introducing some standard notation and defining the variable exponent spaces. In Section 3 we define the trace space and show that it only depends on the value of the exponent on the boundary, provided the exponent is log-Hölder continuous. In Section 4 we derive the formula for the intrinsic norm of the trace space relying on a well-chosen extension of the exponent.

Many open questions still remain regarding trace theory in variable exponent spaces. We consider only extensions from $\mathbb{R}^n$ to the closed half-space $\overline{\mathbb{H}}$. In the fixed exponent case traces have been studied in many other settings than the half-space, see e.g. [17, 18, 22]. Also, we consider only the critical smoothness, $1 - 1/p(x)$. With classical notation, the spaces we consider are the spaces $W^{1-1/p(\cdot),p(\cdot)}(\mathbb{R}^n)$. A future endeavor, then, is to consider also other spaces with variable smoothness on more general domains, i.e. spaces of the type $W^{s(\cdot),p(\cdot)}(\Omega)$.

2. Preliminaries

We will be considering the spaces $\mathbb{H} = \mathbb{R}^n \times (0,\infty)$, its closure $\overline{\mathbb{H}}$ and $\mathbb{R}^n$, which we view as the subspace $\mathbb{R}^n \times \{0\}$ of $\overline{\mathbb{H}}$. An analogous convention holds for arguments of functions, e.g. for $x \in \mathbb{R}^n$ we will sometimes write $p(x)$ instead of $p(x,0)$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B^n(x,r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. By B^n we denote the unit ball $B^n(0,1)$. We use c as a generic constant, i.e. a constant whose values may change from appearance to appearance. By χ_A we denote the characteristic function of the set A. We use the convention that $\chi_A F = 0$ at all points outside A, regardless of whether F is defined there or not.

We denote the mean-value of the integrable function f, defined on a set A of finite non-zero measure, by

$$\langle f\rangle_A = \fint_A f(x)\,dx = \frac{1}{|A|}\int_A f(x)\,dx.$$

For convenience we use a short-hand notation for the average over a ball:

$$\langle f\rangle^n_{x,r} = \langle f\rangle_{B^n(x,r)}.$$

The Hardy-Littlewood maximal operator M is defined on $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{r>0}\, \langle |u|\rangle^n_{x,r}.$$

When there is a possibility of misunderstanding, we will indicate the dimension of the underlying balls, writing $M_{(n)}$ or $M_{(n+1)}$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $p\colon \Omega \to [1,\infty)$ be a measurable bounded function, called a variable exponent on Ω, and denote $p^+ = \operatorname{ess\,sup} p(x)$ and $p^- = \operatorname{ess\,inf} p(x)$. We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $f\colon \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{L^{p(\cdot)}(\Omega)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx$$

is finite. We define the Luxemburg norm on this space by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\big\{\lambda > 0\colon \varrho_{L^{p(\cdot)}(\Omega)}(f/\lambda) \leqslant 1\big\},$$

which is just the Minkowski functional of the absolutely convex set $\{f : \varrho_{L^{p(\cdot)}(\Omega)}(f) \leqslant 1\}$. In the case when $\Omega = \mathbb{R}^n$ we replace the $L^{p(\cdot)}(\mathbb{R}^n)$ in subscripts simply by $p(\cdot)$, i.e. $\|f\|_{p(\cdot)}$ stands for $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, etc. The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the subspace of $L^{p(\cdot)}(\Omega)$ of functions f whose distributional gradient exists and satisfies $|\nabla f| \in L^{p(\cdot)}(\Omega)$. The norm

$$\|f\|_{W^{1,p(\cdot)}(\Omega)} = \|f\|_{L^{p(\cdot)}(\Omega)} + \|\nabla f\|_{L^{p(\cdot)}(\Omega)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

For fixed exponent spaces we of course have a very simple relationship between norm and modular. In the variable exponent case this is not so. However, we nevertheless have the following useful property: $\varrho_{p(\cdot)}(f) \leqslant 1$ if and only if $\|f\|_{p(\cdot)} \leqslant 1$. This and many other basic results were proven in [16, 23].

We say that the exponent p is (*locally*) log-*Hölder continuous* if there exists a constant $c > 0$ such that

$$|p(x) - p(y)| \leqslant \frac{c}{\log(1/|x-y|)}.$$

for all points with $|x - y| < \frac{1}{2}$. Some other names that have been used for these functions are 0-Hölder continuous, Dini-Lipschitz continuous and weak Lipschitz continuous. We say that the exponent p is *globally* log-*Hölder continuous* if it is locally log-Hölder continuous and there exist constants $c > 0$ and $p_\infty \in [1, \infty)$ such that for all points x we have

$$|p(x) - p_\infty| \leqslant \frac{c}{\log(e + |x|)}.$$

Let us denote by $\mathcal{P}(\Omega)$ the class of globally log-Hölder continuous variable exponents on $\Omega \subset \mathbb{R}^n$ with $1 < p^- \leqslant p^+ < \infty$. By [8, Theorem 1.5] we know that

$$M\colon L^{p(\cdot)}(\mathbb{R}^n) \to L^{p(\cdot)}(\mathbb{R}^n) \tag{2.1}$$

is bounded if $p \in \mathcal{P}(\mathbb{R}^n)$. Global log-Hölder continuity is the best possible modulus of continuity to imply the boundedness of the maximal operator, see [8, 27]. But for other, weaker results see [11, 24, 26]. If the maximal operator is bounded, then it follows easily that $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. In general, however, the latter condition is much weaker, see [21].

3. The definition of trace spaces

Recall the definition of the trace of a $W^{1,1}$ function: if $F \in W^{1,1}(\mathbb{H}) \cap C(\overline{\mathbb{H}})$, then $\operatorname{Tr} F = F|_{\mathbb{R}^n}$ and it follows that $\|\operatorname{Tr} F\|_{L^1(\mathbb{R}^n)} \leqslant c\|F\|_{W^{1,1}(\mathbb{H})}$. Having defined a bounded linear operator Tr on a dense subset of $W^{1,1}$ we extend it to all of $W^{1,1}$ continuously.

Let next $F \in W^{1,p(\cdot)}(\mathbb{H})$. Then it follows that $F \in W^{1,1}_{\text{loc}}(\mathbb{H})$. Thus by the previous paragraph $\operatorname{Tr} F$ is defined as a function in $L^1_{\text{loc}}(\mathbb{R}^n)$. Note that if $F \in W^{1,p(\cdot)}(\mathbb{H}) \cap C(\overline{\mathbb{H}})$, then we still have $\operatorname{Tr} F = F|_{\mathbb{R}^n}$. The *trace space* $\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$ consists of the traces of all functions $F \in W^{1,p(\cdot)}(\mathbb{H})$. Notice that the elements of $\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$ are functions defined on $\mathbb{R}^n$ – to emphasize this we will always use lowercase letters for functions on $\mathbb{R}^n$, whereas uppercase letters will be used for functions in $\mathbb{H}$ and $\mathbb{R}^{n+1}$. The quotient norm

$$\|f\|_{\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})} = \inf\left\{\|F\|_{W^{1,p(\cdot)}(\mathbb{H})} \colon F \in W^{1,p(\cdot)}(\mathbb{H}) \text{ and } \operatorname{Tr} F = f\right\}$$

makes $\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$ a Banach space. The main purpose of this paper is to provide an intrinsic norm for the trace space, i.e. a norm which is defined only in terms of f and not in terms of its extension F.

Furthermore, intuitively we would expect that this intrinsic norm only depends on $p|_{\mathbb{R}^n}$ and not on p on the whole space $\overline{\mathbb{H}}$. Nevertheless, the definition of $\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$ above is dependent on the values of p on all of $\mathbb{H}$. It has often been the case that log-Hölder continuity of the exponent p is a sufficient condition for variable exponent spaces to behave in a very nice way. This turns out to hold also with trace spaces:

Theorem 3.1. *Let $p_1, p_2 \in \mathcal{P}(\overline{\mathbb{H}})$ with $p_1|_{\mathbb{R}^n} = p_2|_{\mathbb{R}^n}$ Then $\operatorname{Tr} W^{1,p_1(\cdot)}(\mathbb{H}) = \operatorname{Tr} W^{1,p_2(\cdot)}(\mathbb{H})$ with equivalent norms.*

We will give the proof of this theorem using an extension from $W^{1,p(\cdot)}(\mathbb{H})$ to $W^{1,p(\cdot)}(\mathbb{R}^{n+1})$. In the following proofs we need also the lower half-space $-\mathbb{H} = \mathbb{R}^n \times (-\infty, 0)$, its closure $-\overline{\mathbb{H}}$ and $\mathbb{R}^{n+1}_{\neq 0} = \mathbb{R}^n \times (\mathbb{R} \setminus \{0\})$.

Definition 3.2. Let us call φ a standard mollifier on $\mathbb{R}^{n+1}$ if $\varphi \in C^\infty(\mathbb{R}^{n+1})$ with $\varphi \geqslant 0$, $\int \varphi \, d\xi = 1$ and $\operatorname{supp} \varphi \subset B^{n+1}(0,1)$. We call $\{\varphi_t\}$ a *standard mollifier family* if φ is a standard mollifier and $\varphi_t(\xi) = t^{-n-1}\varphi(\xi/t)$.

Note that if $p \in \mathcal{P}(\mathbb{R}^n)$ and $\{\varphi_i\}$ is a standard mollifier family, then $\varphi_t * f \to f$ in $W^{1,p(\cdot)}(\Omega)$ for all $f \in W^{1,p(\cdot)}(\Omega)$, see [6, 9].

Theorem 3.3. *Let $p \in \mathcal{P}(\mathbb{R}^{n+1})$. Then there exists a bounded, linear extension operator $\mathcal{E} : W^{1,p(\cdot)}(\mathbb{H}) \to W^{1,p(\cdot)}(\mathbb{R}^{n+1})$.*

Proof. Let $F \in W^{1,p(\cdot)}(\mathbb{H})$. It follows from $p \in \mathcal{P}(\mathbb{R}^{n+1})$ that $C_0^\infty(\overline{\mathbb{H}})$ is dense in $W^{1,p(\cdot)}(\mathbb{H})$, hence it suffices to prove the claim for $F \in C^\infty(\mathbb{H})$. Let $\{\varphi_t\}$ be a standard mollifier family on $\mathbb{R}^{n+1}$. Then we define $\mathcal{E}F : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\mathcal{E}F(x,t) := \begin{cases} F(x,t) & \text{for } t \geqslant 0, \\ (\varphi_{|t|} * F)(x,|t|) & \text{for } t < 0 \end{cases}$$

We have to show that $\mathcal{E}F \in W^{1,p(\cdot)}(\mathbb{R}^{n+1})$ with bounded norm. Obviously, $\mathcal{E}F \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{n+1}_{\neq 0})$. In the following we denote $\xi := (x,t) \in -\mathbb{H}$ and $\xi' := (x,|t|)$. We directly estimate

$$\begin{aligned} |(\mathcal{E}F)(\xi)| &= |(\varphi_{|t|} * F)(\xi')| \leqslant c(\varphi)\, M_{(n+1)}(\chi_{\mathbb{H}} F)(\xi'), \\ |\nabla_x(\mathcal{E}F)(\xi)| &= |(\varphi_{|t|} * \nabla_x F)(\xi')| \leqslant c(\varphi)\, M_{(n+1)}(\chi_{\mathbb{H}} \nabla_x F)(\xi'). \end{aligned}$$

For the t-derivative we need a slightly more involved calculation: for all $\xi \equiv (x,t) \in -\mathbb{H}$ and $a \in \mathbb{R}$ we have

$$\begin{aligned} \partial_t(\mathcal{E}F)(\xi) &= \frac{\partial}{\partial t}(\varphi_{|t|} * F)(\xi') = \frac{\partial}{\partial t}\big(\varphi_{|t|} * (F-a)\big)(\xi') \\ &= \int_{\mathbb{R}^{n+1}} \Big[\frac{n}{|t|}\varphi_{|t|}(\xi'-\eta) + \frac{1}{|t|^2}(\nabla_\xi \varphi)_{|t|}(\xi'-\eta)\cdot(\xi'-\eta)\Big]\big(F(\eta)-a\big)\, d\eta, \end{aligned}$$

where $(\nabla\varphi)_r(\eta) := r^{-n-1}\nabla\varphi(\eta/r)$, for $r > 0$. Setting $a = \langle F\rangle^{n+1}_{\xi',|t|}$ we find that

$$\begin{aligned} |\partial_t(\mathcal{E}F)(\xi)| &\leqslant \int_{\mathbb{R}^{n+1}} \Big(\frac{n}{t^{n+2}}\|\varphi\|_\infty + \frac{1}{|t|^{n+3}}\|\nabla_\xi\varphi\|_\infty|\xi'-\eta|\Big)\big|F(\eta) - \langle F\rangle^{n+1}_{\xi',|t|}\big|\, d\eta \\ &\leqslant |t|^{-n-2}\big(n\|\varphi\|_\infty + \|\nabla_\xi\varphi\|_\infty\big)\int_{B^{n+1}(\xi',|t|)} \big|F(y) - \langle F\rangle^{n+1}_{\xi',|t|}\big|\, d\eta \\ &= \frac{c(\varphi)}{t} \fint_{B^{n+1}(\xi',|t|)} \big|F(y) - \langle F\rangle^{n+1}_{\xi',|t|}\big|\, d\eta. \end{aligned}$$

Then the Poincaré inequality implies that

$$|\partial_t(\mathcal{E}F)(\xi)| \leqslant c(\varphi)\,\big\langle|\nabla_\xi F|\big\rangle^{n+1}_{\xi',|t|} \leqslant c(\varphi)\, M_{(n+1)}(\chi_{\mathbb{H}}\nabla_\xi F)(\xi').$$

Overall, we have shown that for all $\xi \equiv (x,t) \in -\mathbb{H}$ that

$$|(\mathcal{E}F)(\xi)| \leqslant c\, M_{(n+1)}(\chi_{\mathbb{H}}F)(\xi'), \quad |\nabla_\xi(\mathcal{E}F)(\xi)| \leqslant c\, M_{(n+1)}(\chi_{\mathbb{H}}\nabla_\xi F)(\xi').$$

Since M is continuous from $L^{p(\cdot)}(\mathbb{R}^{n+1})$ to $L^{p(\cdot)}(\mathbb{R}^{n+1})$, this point-wise inequality implies that

$$\|\mathcal{E}F\|_{W^{1,p(\cdot)}(\mathbb{R}^{n+1}_{\neq 0})} \leqslant c\|MF\|_{L^{p(\cdot)}(\mathbb{H})} + c\|M(\chi_{\mathbb{H}}\nabla F)\|_{L^{p(\cdot)}(\mathbb{H})} \leqslant c\|F\|_{W^{1,p(\cdot)}(\mathbb{H})}.$$

It remains to show that $\mathcal{E}F$ has a distributional gradient in $\mathbb{R}^{n+1}$, which, by [31, Theorem 2.1.4.] follows once we show that $\mathcal{E}F$ is absolutely continuous on lines. Recall that this means (by definition) that the set of values $x \in \mathbb{R}^n$ for which the function $t \mapsto \mathcal{E}F(x,t)$ is not absolutely continuous on $\mathbb{R}$ has n-measure zero, and similarly for all the other co-ordinate directions. We easily see that $\mathcal{E}F \in C(\mathbb{R}^{n+1})$, and that $\mathcal{E}F$ is ACL on both $\mathbb{H}$ and $-\mathbb{H}$, from which it directly follows that $\mathcal{E}F \in ACL(\mathbb{R}^{n+1})$, so $\nabla\mathcal{E}F$ exists. □

We now show how Theorem 3.3 implies Theorem 3.1.

Proof of Theorem 3.1. Define $q(x,t) := p_1(x,t)$ for $t \geqslant 0$ and $q(x,t) := p_2(x,-t)$ for $t < 0$. Then $q \in \mathcal{P}(\mathbb{R}^{n+1})$. By Theorem 3.3 there exist bounded, linear extensions

$$\mathcal{E}_1 : W^{1,q(\cdot)}(\mathbb{H}) \to W^{1,q(\cdot)}(\mathbb{R}^{n+1}),$$
$$\mathcal{E}_2 : W^{1,q(\cdot)}(-\mathbb{H}) \to W^{1,q(\cdot)}(\mathbb{R}^{n+1}).$$

This directly implies $\operatorname{Tr} W^{1,q(\cdot)}(-\mathbb{H}) = \operatorname{Tr} W^{1,q(\cdot)}(\mathbb{H})$ with equivalence of norms. The identities $\operatorname{Tr} W^{1,q(\cdot)}(\mathbb{H}) = \operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$ and $\operatorname{Tr} W^{1,q(\cdot)}(-\mathbb{H}) = \operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$ (by reflection) conclude the proof of the theorem. □

Remark 3.4. The first author has previously proven an extension theorem for variable exponent spaces, see [10, Theorem 4.2]. The difference between that result and Theorem 3.3 is the following: in Theorem 3.3 the exponent p is already given outside of $\overline{\mathbb{H}}$ while in [10] the exponent p had to be extended from $\overline{\mathbb{H}}$ to $\mathbb{R}^{n+1}$ in a special way.

Recall the definition of the Sobolev space of functions with zero boundary value: the space $W_0^{1,p(\cdot)}(\mathbb{H})$ is the completion of $C_0^\infty(\mathbb{H})$ in $W^{1,p(\cdot)}(\mathbb{H})$. (Other definitions are better, when smooth functions are not dense, see [19, 20].) We next characterize $W_0^{1,p(\cdot)}(\mathbb{H})$ in terms of traces.

Theorem 3.5. *Suppose that $p \in \mathcal{P}(\overline{\mathbb{H}})$ and let $F \in W^{1,p(\cdot)}(\mathbb{H})$. Then $F \in W_0^{1,p(\cdot)}(\mathbb{H})$ if and only if $\operatorname{Tr} F = 0$.*

Proof. Suppose first that $F \in W^{1,p(\cdot)}(\mathbb{H})$ with $\operatorname{Tr} F = 0$. We extend p to $-\mathbb{H}$ by reflection. Since $W^{1,p(\cdot)}(\mathbb{H}) \hookrightarrow W^{1,1}_{\text{loc}}(\mathbb{H})$, it follows by classical theory that F extended by 0 to the lower half-space $-\mathbb{H}$ is differentiable in the sense of distributions in $\mathbb{R}^{n+1}$, and hence F is in $W^{1,p(\cdot)}(\mathbb{R}^{n+1})$. Now let φ be a standard mollifier with support in $B^{n+1}(-e_{n+1}/2, 1/3)$. Then $\varphi_r * F$ has compact support in $\mathbb{H}$ and is smooth. Since $p \in \mathcal{P}(\mathbb{R}^{n+1})$, it follows that $\varphi_r * F \to F$ in $W^{1,p(\cdot)}(\mathbb{R}^{n+1})$ as $r \to 0$, so $F \in W_0^{1,p(\cdot)}(\mathbb{H})$.

For the converse, if $F \in W_0^{1,p(\cdot)}(\mathbb{H})$, then, by definition, $F = \lim \varphi_i$ in $W^{1,p(\cdot)}(\mathbb{H})$, where $\varphi_i \in C_0^\infty(\mathbb{H})$. Since $\operatorname{Tr}\varphi_i = \varphi_i|_{\mathbb{R}^n} \equiv 0$, the claim follows by continuity of $\operatorname{Tr} : W^{1,p(\cdot)}(\mathbb{H}) \to \operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$. □

The following simple result was proven recently in [7, Lemma 4.3]. We include the proof for completeness, since our proof is much shorter than that in [7].

Proposition 3.6. *Let $X \subset \mathbb{R}^n$. If $p \in \mathcal{P}(X)$, then there exists an extension $\tilde{p} \in \mathcal{P}(\mathbb{R}^n)$.*

Proof. Since $1/\log(1/t)$ is convex on $(0, 1/8)$, we can use a McShane-type maximal extension [25] of p to $X_{1/8} := \{x \in \mathbb{R}^n : d(x, X) < 1/8\}$, more precisely, we define

$$p(y) = \sup_{x \in X} \Big(p(x) + \frac{C}{-\log|x-y|} \Big),$$

where C is the local log-Hölder constant of p. Iterating this procedure we can define p on all of $\mathbb{R}^n$. Then we define

$$\bar{p}(x) = (1 - d(x, X))p(x) + d(x, X)p_\infty$$

for $d(x, X) < 1$ and $\bar{p}(x) = p_\infty$ otherwise (p_∞ is defined by p if X is unbounded, and can be chosen arbitrarily otherwise). Finally we truncate $\bar{p}$ at p_X^- and p_X^+ to get $\tilde{p}$. □

The previous proposition and Theorem 3.1 imply that the following definition is sensible (up to equivalence of norms).

Definition 3.7. Let $p \in \mathcal{P}(\mathbb{R}^n)$ and let $q \in \mathcal{P}(\mathbb{H})$ be an arbitrary extension of p. Then we define an intrinsic trace space by

$$\big(\operatorname{Tr} W^{1,p(\cdot)}\big)(\mathbb{R}^n) := \operatorname{Tr} W^{1,q(\cdot)}(\mathbb{H}).$$

Remark 3.8. When $p \in \mathcal{P}(\mathbb{R}^n)$, Theorem 3.1 simplifies studying the space $(\operatorname{Tr} W^{1,p(\cdot)})(\mathbb{R}^n)$ significantly. Indeed, for $x \in \mathbb{R}^n$ and $t \in [0, 2]$ define $q(x, t) := p(x)$. Then q is globally log-Hölder continuous on $\mathbb{R}^n \times [0, 2]$ with $1 < q^- \leqslant q^+ < \infty$. By Proposition 3.6 we can extend q to the set $\overline{\mathbb{H}}$ so that $q \in \mathcal{P}(\overline{\mathbb{H}})$. We have $(\operatorname{Tr} W^{1,p(\cdot)})(\mathbb{R}^n) = \operatorname{Tr} W^{1,q(\cdot)}(\mathbb{H})$. So we can always assume that the exponent $q(x, t)$ is independent of t as long as $t \in [0, 2]$.

4. Intrinsic Characterization of the Trace Space

For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we define the sharp operator by

$$M^\sharp_{B^n(x,r)} f = \fint_{B^n(x,r)} \big|f(y) - \langle f\rangle^n_{x,r}\big|\, dy,$$

Using the triangle inequality it is easy to show that

$$M^\sharp_{B^n(x,r)} f \leqslant \fint_{B^n(x,r)} \fint_{B^n(x,r)} \big|f(y) - f(z)\big|\, dy\, dz \leqslant 2\, M^\sharp_{B^n(x,r)} f. \tag{4.1}$$

We define the *trace modular* $\varrho_{\mathrm{Tr},p(\cdot)}$ by

$$\varrho_{\mathrm{Tr},p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx + \int_0^1 \int_{\mathbb{R}^n} \Big(\tfrac{1}{r} M^\sharp_{B^n(x,r)} f\Big)^{p(x)} dx\, dr.$$

Obviously, $\varrho_{\mathrm{Tr},p(\cdot)}$ is convex. Thus

$$\|f\|_{\mathrm{Tr},p(\cdot)} := \inf\big\{\lambda > 0 :\ \varrho_{\mathrm{Tr},p(\cdot)}(f/\lambda) \leqslant 1\big\}$$

is a norm, since it is the Minkowski functional of the absolutely convex set $\{f\ :\ \varrho_{\mathrm{Tr},p(\cdot)}(f) \leqslant 1\}$.

The following theorem characterizes the traces of $W^{1,p(\cdot)}(\mathbb{H})$ functions and completes the proof of Theorem 1.1.

Theorem 4.2. *Let $p \in \mathcal{P}(\overline{\mathbb{H}})$ and let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then f belongs to $\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$ if and only if $\|f\|_{\mathrm{Tr},p(\cdot)} < \infty$, or, equivalently,*

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx + \int_0^1 \int_{\mathbb{R}^n} \Big(\tfrac{1}{r} M^\sharp_{B^n(x,r)} f\Big)^{p(x)} dx\, dr < \infty,$$

where $p(x) := p(x,0)$. Moreover, $\|f\|_{\mathrm{Tr},p(\cdot)}$ is equivalent to the quotient norm $\|f\|_{\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})}$.

To prove the theorem we have to show two things. First, for $F \in W^{1,p(\cdot)}(\mathbb{H})$ and $f := \operatorname{Tr} F$ we have to show that $\|f\|_{\mathrm{Tr},p(\cdot)} \leqslant c\,\|F\|_{\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})}$. Therefore, we have to estimate $|f|$ and $M^\sharp_{B^n(x,t)} f$ in terms $|F|$ and $|\nabla F|$. Second, for $f \in \operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$ we have to show the existence of some $F \in W^{1,p(\cdot)}(\mathbb{H})$ with $\operatorname{Tr} F = f$ and $\|F\|_{\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})} \leqslant c\,\|f\|_{\mathrm{Tr},p(\cdot)}$. We will define the extension F by $F(x,t) := (\varphi_t * f)(x)$ for $x \in \mathbb{R}^n$ and $t > 0$, where (φ_t) is a standard mollifier family. In order to estimate $\|F\|_{W^{1,p(\cdot)}(\mathbb{H})}$ we need to estimate $|F|$ and $|\nabla F|$ in terms of $|f|$ and $M^\sharp_{B^n(x,t)} f$. The following two lemmas provide these estimates.

Lemma 4.3. *There exists a constant $c_1 > 0$, such that*

$$M^\sharp_{B^n(z,r)} \operatorname{Tr} F \leqslant c_1\, r \fint_{B^{n+1}((z,0),r)} \chi_{\mathbb{H}}(\xi)\, |\nabla F(\xi)|\, d\xi \tag{4.4}$$

for all $z \in \mathbb{R}^n$, $r > 0$ and $F \in W^{1,1}\big(B^{n+1}((z,0),r)\big)$.

Proof. Since smooth functions are dense in $W^{1,1}(B^{n+1}(z,r))$ it suffices to prove (4.4) for smooth F. As usual we denote $f = \operatorname{Tr} F = F|_{\mathbb{R}^n}$. Let us estimate $|f(x) - f(y)|$ for $x, y \in \mathbb{R}^n$ by integrating the gradient over the path $\gamma_\zeta = [x,\zeta] \cup [\zeta, y]$ for $\zeta \in \mathbb{H}$:

$$|f(x) - f(y)| \leqslant \int_{\gamma_\zeta} |\nabla F(\xi)|\, d\xi. \tag{4.5}$$

Define $B_{x,y} = B^{n+1}(\frac{x+y}{2} + \frac{|x-y|}{4} e_{n+1}, \frac{|x-y|}{8}) \cap P$, where P is the mid-point normal plane of the segment $[x,y]$. We integrate (4.5) over $\zeta \in B_{x,y}$ and get

$$|f(x) - f(y)| \leqslant c \int_{A_{x,y}} |\nabla F(\xi)| \big(|x-\xi|^{-n} + |y-\xi|^{-n}\big)\, d\xi,$$

where $A_{x,y} = \bigcup_{\zeta \in B_{x,y}} \gamma_\zeta$. Let $z \in \mathbb{R}^n$ and $r > 0$. Using the previous estimate together with (4.1) gives

$$M^\sharp_{B^n(z,r)} f \leqslant c \fint_{B^n(z,r)} \fint_{B^n(z,r)} \int_{A_{x,y}} |\nabla F(\xi)| t^{-n}\, d\xi\, dx\, dy, \tag{4.6}$$

where t denotes the $n+1^{\text{st}}$ co-ordinate of ξ and we used that $t \leqslant \min\{|y-\xi|, |x-\xi|\}$ when $\xi \in A_{x,y}$.

The set $A_{x,y}$ consists of two cones, one emanating from y and the other from x, denoted by $A'_{x,y}$ and $A''_{x,y}$, respectively. By symmetry, we see that we can replace $A_{x,y}$ by $A'_{x,y}$ in (4.6). We want to swap the order of integration. So suppose that $\xi \in A'_{x,y}$. Then certainly $\xi \in B^{n+1}(z,r)$. Also, ξ lies in a cone emanating from y whose direction depends on $x - y$. Thus we see that y lies in the cone emanating from ξ with the same base-angle but opposite direction. This means that for a fixed ξ the variable y varies in a ball $B^n(w, c't)$ for some $w \in B^{n+1}(z,r)$ (depending on $x-y$) and $c' > 0$ (depending only on the dimension n). Hence

$$\begin{aligned} M^\sharp_{B^n(z,r)} f &\leqslant c\, r^{-2n} \int_{B^n(z,r)} \int_{B^n(z,r)} \int_{A'_{x,y}} |\nabla F(\xi)| t^{-n} \, d\xi \, dx \, dy \\ &\leqslant c\, r^{-2n} \int_{B^{n+1}((z,0),r)} \chi_{\mathbb{H}}(\xi) \, |\nabla F(\xi)| t^{-n} \int_{B^n(w,c't)} \int_{B^n(z,r)} dx \, dy \, d\xi \\ &= c\, r \fint_{B^{n+1}(z,r)} \chi_{\mathbb{H}}(\xi) \, |\nabla F(\xi)| \, d\xi. \end{aligned}$$

This proves the lemma. □

The proof of the following lemma is essentially the same as the proof of Theorem 3.3, so it is omitted here.

Lemma 4.7. *Let $\{\varphi_t\}$ be a standard mollifier family. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and define $F(x,t) := (\varphi_t * u)(x)$ for $x \in \mathbb{R}^n$ and $t \in (0,\infty)$. Then there exists a constant c_2 depending only on φ and n such that for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$*

$$\begin{aligned} |F(x,t)| &\leqslant c_2 \langle |f| \rangle^n_{x,t}, \\ |\nabla F(x,t)| &\leqslant \frac{c_2}{t} \, M^\sharp_{B^n(x,t)} f. \end{aligned}$$

Thus we are ready for the proof of the main result.

Proof of Theorem 4.2. Due to Theorem 3.1 and Remark 3.8 we can assume without loss of generality that $p(x,t) = p(x,0) = p(x)$ for $x \in \mathbb{R}^n$ and $t \in [0,2]$.

Let $\{\varphi_t\}$ be a standard mollifier family, and let $f \in \operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})$ with $\|f\|_{\operatorname{Tr} W^{1,p(\cdot)}(\mathbb{H})} \leqslant 1$, equivalently, with

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx + \int_0^1 \int_{\mathbb{R}^n} \Big(\tfrac{1}{r} M^\sharp_{B^n(x,r)} f \Big)^{p(x)} dx \, dr \leqslant 1. \tag{4.8}$$

We have to show the existence of an extension $F \in W^{1,p(\cdot)}(\mathbb{H})$ with $\|F\|_{W^{1,p(\cdot)}(\mathbb{H})} \leqslant c$, where c is independent of f. As mentioned above, we would like to consider the extension $(x,t) \mapsto (\varphi_t * f)(x)$. But in order to avoid difficulties as $t \to \infty$ we cut off the part for large t. Let $\psi \in C_0^\infty([0,\infty))$ with $\chi_{B^1(0,1/2)} \leqslant \psi \leqslant \chi_{B^1(0,1)}$. Then our extension F is given by $F(x,t) := (\varphi_t * f)(x) \, \psi(t)$.

We now estimate the norm of F in $W^{1,p(\cdot)}(\mathbb{H})$. Using Lemma 4.7 and noting that $\langle |f| \rangle^n_{x,t} \leqslant Mf(x)$, we find that

$$\varrho_{L^{p(\cdot)}(\mathbb{H})}(F) = \int_0^1 \int_{\mathbb{R}^n} |F(x,t)|^{p(x)} \, dx \, dt \leqslant c \int_{\mathbb{R}^n} \big(Mf(x) \big)^{p(x)} \, dx.$$

Our assumptions on p imply that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Since $\varrho_{p(\cdot)}(f) \leqslant 1$, the previous inequality implies that $\varrho_{L^{p(\cdot)}(\mathbb{H})}(F) \leqslant c$. We move to the norm of the gradient. Using Lemma 4.7 again, we estimate

$$\varrho_{L^{p(\cdot)}(\mathbb{H})}(\nabla F) = \int_0^1 \int_{\mathbb{R}^n} |\nabla F(x,t)|^{p(x)} \, dx \, dt \leqslant c(\psi) \int_0^1 \int_{\mathbb{R}^n} \big| \tfrac{1}{t} \, M^\sharp_{B^n(x,t)} f \big|^{p(x)} \, dx \, dt \leqslant c.$$

Thus we have shown that $F \in W^{1,p(\cdot)}(\mathbb{H})$. Furthermore, it follows easily that $f = \operatorname{Tr} F$, so we have proved one of the implications in the theorem.

To prove the opposite implication, we use the density of smooth functions and restrict ourselves without loss of generality to $F \in W^{1,p(\cdot)}(\mathbb{H}) \cap C^\infty(\overline{\mathbb{H}})$. By homogeneity, it suffices to consider the case $\|F\|_{W^{1,p(\cdot)}(\mathbb{H})} \leqslant 1$ and to prove $\|f\|_{\operatorname{Tr},p(\cdot)} \leqslant C$. Since p is bounded, the latter condition is equivalent to $\varrho_{\operatorname{Tr},p(\cdot)}(f) \leqslant C$, which is what we now prove. Replacing F by $F\,\psi$, where ψ is as above, we see that it suffices to consider F supported in $\mathbb{R}^n \times [0,1]$. Define $f := \operatorname{Tr} F$. We find that

$$|f(x)| = |F(x,0)| \leqslant \int_0^1 |\nabla F(x,t)|\,dt.$$

Hence using Jensen's inequality we get that

$$|f(x)|^{p(x)} \leqslant \int_0^1 |\nabla F(x,t)|^{p(x)}\,dt,$$

and, integrating over $x \in \mathbb{R}^n$,

$$\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \leqslant \int_{\mathbb{R}^n}\int_0^1 |\nabla F(x,t)|^{p(x)}\,dt\,dx = \varrho_{L^{p(\cdot)}(\mathbb{H})}(\nabla F).$$

Thus we have bounded the $L^{p(\cdot)}$ part of the trace norm.

Since $f = \operatorname{Tr} F$, we get by Lemma 4.3 that

$$\begin{aligned}
\int_0^1\int_{\mathbb{R}^n} \Big(\tfrac{1}{r} M^\sharp_{B^n(x,r)} f\Big)^{p(x)}\,dx\,dr &\leqslant c\int_0^1\int_{\mathbb{R}^n}\bigg(\fint_{B^{n+1}((x,0),r)} \chi_{\mathbb{H}}(\xi)\,|\nabla F(\xi)|\,d\xi\bigg)^{p(x)}\,dx\,dr\\
&\leqslant c\int_0^1\int_{\mathbb{R}^n}\bigg(\fint_{B^{n+1}((x,r),2r)} \chi_{\mathbb{H}}(\xi)\,|\nabla F(\xi)|\,d\xi\bigg)^{p(x)}\,dx\,dr\\
&\leqslant c\int_{\mathbb{R}^n\times[0,1]} \Big(M_{(n+1)}\big(\chi_{\mathbb{H}}\,|\nabla F|\big)(\eta)\Big)^{p(\eta)}\,d\eta.
\end{aligned}$$

Extending the exponent to the lower half-space by reflection, we immediately see that $p \in \mathcal{P}(\mathbb{R}^{n+1})$ and

$$\int_0^1\int_{\mathbb{R}^n} \Big(\tfrac{1}{r} M^\sharp_{B^n(x,r)} f\Big)^{p(x)}\,dx\,dr \leqslant c\int_{\mathbb{R}^{n+1}} \Big(M_{(n+1)}\big(\chi_{\mathbb{H}}\,|\nabla F|\big)(\xi)\Big)^{p(\xi)}\,d\xi.$$

Since the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^{n+1})$ the right-hand-side of the previous inequality is bounded by a constant depending only on $\|\nabla F\|_{p(\cdot)} \leqslant 1$. This concludes the proof. □

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