Exercise: The purpose of this exercise is to prove sharp L^q bounds for eigenfunctions of the Laplacian.

Theorem 0.1 (Sogge 1988)**.** *Let (M,g) be a compact n-dimensional Riemannian manifold and $-\Delta_g$ the Laplace-Beltrami operator on M. For any $q \in [2,\infty]$ there exists $C_q > 0$ such that*

$$\|\mathbf{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^q(M)} \le C_q(1+\lambda)^{\delta(q)}, \tag{1}$$

where

$$\delta(q) := \begin{cases} \frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right), & 2 \le q \le q_n := \frac{2(n+1)}{n-1} \\ n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2}, & q_n \le q \le \infty. \end{cases}$$

(a) Observe that the endpoint $q=2$ is trivial and that we have already proved the result for $q=\infty$ in the lecture. Conclude (by interpolation) that it suffices to prove the estimate for $q=q_n$.

(b) First consider the Euclidean case $g^{ij}=\delta_i^j$. Fix $\chi \in \mathcal{S}(\mathbb{R})$. Using stationary phase, show that the Schwartz kernel

$$\left(\chi(\sqrt{-\Delta_{\mathbb{R}^n}}-\lambda)\right)(x,y) = \int_{\mathbb{R}^n} \chi(|\xi|-\lambda)\mathrm{e}^{\mathrm{i}\xi\cdot(x-y)}\mathrm{d}\xi \tag{2}$$

satisfies

$$\left|\left(\chi(\sqrt{-\Delta_{\mathbb{R}^n}}-\lambda)\right)(x,y)\right| \lesssim \lambda^{n-1}(1+\lambda|x-y|)^{-\frac{n-1}{2}}. \tag{3}$$

(c) Let Γ be a conic cutoff to a small neighborhood of $e_1 \in S^{n-1}$. Define K to be the kernel in (2) with an additional $\Gamma(\xi)$ inserted in the integral and, for fixed x_1, y_1, let K_{x_1,y_1} be the corresponding kernel in $2(n-1)$ variables (x',y'). By slight abuse of notation we denote the operators corresponding to K and K_{x_1,y_1} by the same symbols. Prove that

$$\|K_{x_1,y_1}\|_{L^1(\mathbb{R}^{n-1})\to L^\infty(\mathbb{R}^{n-1})} \lesssim \lambda^{n-1}(1+\lambda|x_1-y_1|)^{-\frac{n-1}{2}} \tag{4}$$

and

$$\|K_{x_1,y_1}\|_{L^2(\mathbb{R}^{n-1})\to L^2(\mathbb{R}^{n-1})} \lesssim 1. \tag{5}$$

Hint: For the first estimate use (3). For the second use Plancherel in the x'-variables.

(d) Interpolate between (4) and (5) to obtain

$$\|K_{x_1,y_1}\|_{L^{q'}(\mathbb{R}^{n-1})\to L^q(\mathbb{R}^{n-1})} \lesssim \lambda^{(n-1)(1-2/q)}(1+\lambda|x_1-y_1|)^{-\frac{n-1}{2}(1-2/q)} \tag{6}$$

for all $q \in [2,\infty]$. Combine this with the Hardy-Littlewood-Sobolev inequality in the x_1-variable to deduce the estimate

$$\|K\|_{L^{q_n'}(\mathbb{R}^n)\to L^{q_n}(\mathbb{R}^n)} \lesssim \lambda^{2\delta(q_n)}. \tag{7}$$

By a partition of unity of the unit sphere S^{n-1} derive the same estimate for $\chi(\sqrt{-\Delta_{\mathbb{R}^n}}-\lambda)$.

(e) We return to a compact manifold M with metric g. Repeat the orthogonality argument shown in the lecture to prove that (1) would follow from

$$\|\chi(\sqrt{-\Delta_g}-\lambda)\|_{L^2(M)\to L^{q_n}(M)} \lesssim \lambda^{2\delta(q_n)}, \tag{8}$$

where $\chi \in \mathcal{S}(\mathbb{R})$, $\chi(0)=1$ and $\widehat{\chi}$ is supported in $[-\delta,\delta]$.

(f) Use the Fourier inversion formula to write

$$\chi(\sqrt{-\Delta_g}-\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda}\widehat{\chi}(t)\mathrm{e}^{\mathrm{i}t\sqrt{-\Delta_g}}\mathrm{d}t. \tag{9}$$

As in the lecture, argue that modulo rapidly decaying tails in λ one can replace $\mathrm{e}^{\mathrm{i}t\sqrt{-\Delta_g}}$ by $\cos(t\sqrt{-\Delta_g})$ for the purpose of proving (8).

(g) Using the Hadamard parametrix for $\cos(t\sqrt{-\Delta_g})$ and (9), use similar arguments as in the Euclidean case to prove (8). Hint: You may find int convenient to use the fact that, modulo smooth functions in (t,x), the distributions $E_\nu(t,x)$ appearing in the parametrix are linear combinations of terms of the form

$$H(t)t^j \int\limits_{|\xi|\geq 1} \mathrm{e}^{\mathrm{i}x\cdot\xi \pm \mathrm{i}t|\xi|}|\xi|^{-\nu-1-k}\mathrm{d}\xi \quad (j+k=\nu).$$