Exercise (Hadamard parametrix for the resolvent): Consider the "Bessel potential" on $\mathbb{R}^n$ (in the distributional sense)

$$F_0(|x|) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}(|\xi|^2 - z)^{-1}\mathrm{d}\xi, \qquad (1)$$

where $z \in \mathbb{C} \setminus \mathbb{R}$ is fixed.

(a) Show that $F_0(|x|)$ is a fundamental solution to $(-\Delta - z)$. More generally, assuming that the matrix (g^{jk}) is constant, show that

$$\left(-\partial_j g^{jk}\partial_k - z\right) F_0(|x|_g) = (\det g^{jk})^{1/2}\delta_0(x). \qquad (2)$$

(b) Consider a second order partial differential operator

$$P(x, D) = -\partial_j g^{jk}(x)\partial_k + b_j(x)\partial_j + c(x), \qquad (3)$$

where the coefficients g^{jk}, b_j and c are smooth on an open set $\Omega \subset \mathbb{R}^n$ and (g^{jk}) is a real positive definite matrix. Choose geodesic normal coordinates near 0 vanishing there and recall that in these coordinates

$$g^{jk}(x)x_k = g^{jk}(0)x_k \quad j = 1, \ldots, n. \qquad (4)$$

Prove that, if $\eta \in C_0^\infty(\mathbb{R}^n)$ is supported in a small enough neighborhood V of zero, then

$$(P(x, D) - z)\,\eta(x)F_0(|x|_g) = \eta(0)(\det g^{jk}(0))^{1/2}\delta_0(x) + R(x, D)F_0(|x|_g), \qquad (5)$$

where $R(x, D)$ is a first order differential operator whose coefficients are supported in V and independent of z. By solving a transport equation, show that η can be chosen in such a way that the first order terms of $R(x, D)$ vanish.

(c) Consider now

$$F_\nu(|x|) = \nu!(2\pi)^{-n} \int\limits_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}(|\xi|^2 - z)^{-\nu-1}\mathrm{d}\xi, \quad \nu = 1, 2, \ldots. \tag{6}$$

Show that

$$(-\Delta - z)F_\nu(|x|) = \nu F_{\nu-1}(|x|), \quad -2\partial_x F_\nu(|x|) = xF_{\nu-1}(|x|)$$

and drive the analogue identity to (2) and (5).

(d) Let Y be a fixed relatively compact subset of Ω. By choosing geodesic normal coordinates near an arbitrary point of $y \in Y$ find a right parametrix E_N of $(P(x, D) - z)$ with Schwartz kernel of the form

$$E_N(x, y) = \sum_{\nu=0}^{N} \eta_\nu(x, y)F_\nu(d_g(x, y))$$

in a neighborhood of the diagonal of $Y \times Y$ and satisfying

$$(P(x, D) - z)\, E_N(x, y) = (\det g^{jk}(y))^{1/2}\delta_y(x) + R_N(x, y), \tag{7}$$

where $R_N \in C^{2N+1-n}$ and smooth away from the diagonal.

(e) On the level of operators (7) means

$$(P(x, D) - z)\, E_N = I + R_N, \tag{8}$$

where R_N is the operator with kernel $R_N(x, y)$. By taking the adjoint of (8) find a left parametrix, i.e. an operator $\widetilde{E}_N$ such that

$$\widetilde{E}_N\, (P(x, D) - z) = I + \widetilde{R}_N, \tag{9}$$

where $\widetilde{R}_N$ has similar smoothing properties as R_N.