Problem Sheet 6

Ex. 3 By duality and interpolation (see remark at the end) it suffices to prove

$$T := \langle x \rangle^{-2N} p(x,D) \langle x \rangle^{2N} : L^2 \to L^2 \text{ bdd.}$$

for $N \in \mathbb{N}_0$. We have $\langle x \rangle^{2N} f = \mathcal{F}^{-1}(\langle D_\xi \rangle^{2N} \hat{f})$

$$\Rightarrow Tf(x) = \int e^{ix\cdot\xi} \frac{p(x,\xi)}{\langle x \rangle^{2N}} \langle D_\xi \rangle^{2N} \hat{f}(\xi)\, d\xi$$

$$\overset{\text{IBP}}{=} \int \left[\langle D_\xi \rangle^{2N} \left(e^{ix\cdot\xi} \frac{p(x,\xi)}{\langle x \rangle^{2N}} \right) \right] \hat{f}(\xi)\, d\xi$$

By Leibniz' rule,

$$\langle D_\xi \rangle^{2N} \left(e^{ix\cdot\xi} \frac{p(x,\xi)}{\langle x \rangle^{2N}} \right) \in S^0_{1,0}$$

$$\Rightarrow \|Tf\|_{L^2} \leq C \|f\|_{L^2}.$$

Remark: • By duality, T is L^2-bdd. for $N \in \mathbb{Z}$ since for $N \in \mathbb{Z}_- \Rightarrow T^* = \langle x \rangle^{2N} p^*(x,D) \langle x \rangle^{-2N}$ is of the same form as in the proof above.

• By interpolation, we get that $\langle x \rangle^{-\sigma} p(x,D) \langle x \rangle^{\sigma}$ is L^2-bdd. for all $\sigma \in \mathbb{R}$. The interpolation argument is non-trivial (Stein's complex interpolation theorem).

Ex. 5 ! The claim is generally not true !

We have to assume $m \leq 1$ (or else assume that p is elliptic).

Proof for $m \leq 1$: We want to prove that

$$\mathrm{Dom}(\bar{P}) = \{u \in L^2 : P(x,D)u \in L^2\}$$

"$\subseteq$": By def. of the closure, for $u \in \mathrm{Dom}(\bar{P})$ there ex. $(u_n)_n \subset \mathcal{S}$ s.t. $u_n \xrightarrow{L^2} u$ and (Pu_n) converges in L^2. Then it also converges in $\mathcal{S}'$. Hence $Pu_n \xrightarrow{\mathcal{S}'} p(x,D)u$ (by continuity of $p(x,D): \mathcal{S}' \to \mathcal{S}'$). This means that $p(x,D)u \in L^2$.

"$\supseteq$" We have to show: $\forall u \in L^2$ s.t. $P(x,D)u \in L^2$ $\exists (u_n)_n \subset \mathcal{S}$ s.t. $u_n \xrightarrow{L^2} u$ and $Pu_n \xrightarrow{L^2} p(x,D)u$.

Let $\chi_\varepsilon \in \mathcal{S}$ be as in Ex. 4 and set $u_{\varepsilon_n} := \chi_{\varepsilon_n}(x,D)u$, for a seq. $\varepsilon_n \to 0$ (we just write u_ε). By Ex. 4 part 5: $u_\varepsilon \xrightarrow{L^2} u$. Remains to show:

$$(*) \qquad p(x,D)u_\varepsilon \xrightarrow{L^2} p(x,D)u.$$

Write $p(x,D)u_\varepsilon = \chi_\varepsilon(x,D)\,p(x,D)u + [p(x,D), \chi_\varepsilon(x,D)]u$

Then $\chi_\varepsilon(x,D)\,p(x,D)u \xrightarrow{L^2} p(x,D)u$. Remains to show:

$$(**) \qquad [p(x,D), \chi_\varepsilon(x,D)]u \xrightarrow{L^2} 0.$$

It is suff. to prove (FA 1):

i) $(**)$ holds for $u \in \mathcal{S}$

ii) $\|[p(x,D), \chi_\varepsilon(x,D)]u\|_{L^2} \leq C\|u\|_{L^2}$ (C-indep. of ε)

Pf of i): We have to show: $\forall u \in \mathcal{S}$

$$[p(x,D), \chi_\varepsilon(x,D) - 1]\, u \xrightarrow{L^2} 0$$

(note: the "-1" is for free, i.e. does not change the commutator). We have

$$|\chi_\varepsilon - 1|_k^{(0)} \xrightarrow{\varepsilon \to 0} 0 \quad \forall k \in \mathbb{N}_0 .$$

The claim follows by continuity of the map

$$\begin{aligned} \mathcal{S} \times S^0 &\longrightarrow \mathcal{S} \\ (u, a) &\longmapsto a(x,D)\, u \end{aligned}$$

Pf of ii): Follows by Ex. 4 part I and the L^2-bdd'ness of ΨDO's of order zero ($m \leq 1$!).