Problem Sheet 9

To be discussed on Wednesday, 22.6.2016.

Ex. 1: Prove the following remark from the lecture: If $a \in S^m$ is elliptic at $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus \{0\}$, then there exists $b \in S^{-m}$ and a conic neighborhood Γ of (x_0, ξ_0) such that $a\#b - 1 \in S^{-\infty}_{\mathrm{loc}}(\Gamma)$.

Ex. 2: *Characterization of the wave front set:* Let $u \in \mathcal{S}'$ and $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus \{0\}$. Prove that $(x_0, \xi_0) \notin WF(u)$ if and only if there exists $\varphi \in C_0^\infty$ with $\varphi = 1$ near x_0 and a conic neighborhood Γ of ξ_0 such that $\lambda^k \widehat{\varphi u}$ is bounded in Γ for all $k \in \mathbb{Z}_+$.

Ex. 3: (a) Let $n = 1$. For $\psi \in \mathcal{S}(\mathbb{R})$ define

$$\left\langle \mathrm{p.v.}\frac{1}{\mathrm{x}}, \psi \right\rangle = \lim_{\epsilon \to 0+} \int_{|x| \geq \epsilon} \frac{\psi(x)}{x} dx, \quad \left\langle \frac{1}{\mathrm{x} \pm \mathrm{i}\, 0}, \psi \right\rangle \quad = \lim_{\epsilon \to 0+} \int \frac{\psi(x)}{x \pm \mathrm{i}\, \epsilon} dx.$$

Show that these formulas define tempered distributions satisfying

$$\frac{1}{x \pm \mathrm{i}\, 0} = p.v.\frac{1}{x} \mp \mathrm{i}\, \pi \delta_0$$

(b) Determine the wave front sets of the distributions defined in (a).

Ex. 4: Let V be a subspace of $\mathbb{R}^n$ and $u = u_0 dS$, where $u_0 \in C_0^\infty(V)$ and dS is Lebesgue measure on V. Prove that

$$WF(u) = \mathrm{supp}(u) \times (V^\perp \setminus \{0\}).$$

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