Problem Sheet 8

To be discussed on Wednesday, 15.6.2016.

Ex. 1: Recall that $a(x,D)$ is said to have the local property if $\operatorname{supp}(a(x,D)u) \subset \operatorname{supp}(u)$ for all $u \in \mathcal{S}'$. The goal of this exercise is to show that differential operators are the only pseudodifferential operators that have the local property.

(a) Let $a \in S^{-n}$, and assume that $a(x,D)$ has the local property. Let $\psi \in \mathcal{S}$, $x_0 \in \mathbb{R}$, $\varphi \in C_0^\infty$ such that $\varphi = 1$ in $B_1(0)$, and for $k \in \mathbb{Z}_+$ set $\psi_k(x) := (1-\varphi(k(x-x_0)))\psi(x)$. Prove that $\|a(x,D)(\psi-\psi_k)\|_\infty \le C\|\psi-\psi_k\|_2$ for a constant C independent of $k \in \mathbb{Z}_+$. Use this to prove that $a(x,D)\psi = 0$.

(b) Let $a \in S^m$ for some $m < k \in \mathbb{Z}_+$ such that $a(x,D)$ has the local property. Show by induction that for any $\alpha, \beta \in \mathbb{Z}_+^n$, $(\partial_x^\alpha \partial_\xi^\beta a)(x,D)$ has the local property. Similyrly, show that $a^*(x,D)$ has the local property. Prove that

$$a(x,D)\psi(x) = \sum_{|\alpha|<k+n} \frac{1}{\alpha} \partial_\xi^\alpha a(x,0) D^\alpha \psi(x).$$

Conclude that the symbol a is a polynomial in ξ, i.e. $a(x,D)$ is a differential operator.

Ex. 2: The aim of this exercise is to give an alternative proof of L^2-boundedness of pseudodifferential operators.

(a) Let $\chi_\xi(x) = \mathrm{e}^{\mathrm{i}\,x\cdot\xi} \prod_{j=1}^n (1+\mathrm{i}\,x_j)^{-1}$, and for $\psi \in \mathcal{S}$,

$$\Psi(x,\xi) = \int \chi_\xi(x-y)\psi(y)dy.$$

Show that for $\alpha \in \{0,1\}^n$, $\partial_x^\alpha(\mathrm{e}^{-\mathrm{i}\,x\cdot\xi}\,\Psi) \in L^2(\mathbb{R}^{2n})$ with norm equal to $2^{-(n+|\alpha|)/2}(2\pi)^n\|\psi\|_{L^2(\mathbb{R}^n)}$, and that

$$\left(\prod_{j=1}^n (1+\partial_{\xi_j})\right)\Psi(x,\xi) = \mathrm{e}^{\mathrm{i}\,x\cdot\xi}\,\widehat{\psi}(\xi).$$

Similarly, for $\varphi \in \mathcal{S}$, show that the function

$$\Phi(x,\xi) = (2\pi\,\mathrm{i})^{-n} \int \chi_{-\xi}(\xi-\eta)\widehat{\varphi}(\eta)d\eta$$

satisfies $\Phi \in L^2(\mathbb{R}^{2n})$ with $\|\Phi\|_{L^2(\mathbb{R}^{2n})} = \pi^{n/2}\|\varphi\|_{L^2(\mathbb{R}^n)}$ and

$$\left(\prod_{j=1}^{n}(1-\partial_{x_j})\right)\Phi(x,\xi) = \mathrm{e}^{-\mathrm{i}\,x\cdot\xi}\,\varphi(x).$$

(b) Let $b \in C^{2n}(\mathbb{R}^{2n})$ be such that $\partial_x^\alpha\partial_\xi^\beta b$ is bounded for all $\alpha,\beta \in \{0,1\}^n$ (any symbol of order zero satisfies this condition, but also symbols in the larger class $S^0_{\rho,0}$, $\rho \geq 0$). For $\varphi,\psi \in \mathcal{S}$ set

$$(b(x,D)\psi,\varphi) = (2\pi)^{-n}\int \mathrm{e}^{\mathrm{i}\,x\cdot\xi}\,b(x,\xi)\widehat{\psi}(\xi)\overline{\varphi(x)}dxd\xi.$$

Prove that

$$\begin{aligned}&(2\pi)^n(b(x,D)\psi,\varphi)\\ &= \sum_{\alpha,\beta,\gamma\in\{0,1\}^n}(-1)^{|\beta|}\binom{\gamma}{\alpha}\int \overline{\Phi(x,\xi)}(\partial_x^\alpha\partial_\xi^\beta b(x,\xi))\partial_x^{\gamma-\alpha}(\mathrm{e}^{-\mathrm{i}\,x\cdot\xi}\,\Psi(x,\xi))dxd\xi\end{aligned}$$

and conclude that $b(x,D)$ is bounded on $L^2(\mathbb{R}^n)$.

Jean-Claude Cuenin