Problem Sheet 7

To be discussed on Wednesday, 8.6.2016.

Ex. 1: Let $\Omega \subseteq \mathbb{R}^n$ be open, $m, s \in \mathbb{R}$. We write $f \in H^s_{\mathrm{loc}}(\Omega) \iff \varphi f \in H^s(\mathbb{R}^n)$ for every $\varphi \in C_c^\infty(\Omega)$. Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $u \in H^{-\infty}_{\mathrm{loc}}(\Omega)$ and $f \in H^s_{\mathrm{loc}}(\Omega)$ be such that $p(x, D)u = f$. Prove that $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$.

Ex. 2: Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ be elliptic. Let $u \in H^{-\infty}(\mathbb{R}^n)$ and $f \in H^s(\mathbb{R}^n)$ be such that $p(x, D)u = f$. Prove that for any $s, \sigma \in \mathbb{R}$, there exists a constant $C_{s,\sigma} > 0$, independent of u, such that

$$\|u\|_{H^{s+m}(\mathbb{R}^n)} \leq C_{s\sigma} \left(\|f\|_{H^s(\mathbb{R}^n)} + \|u\|_{H^\sigma(\mathbb{R}^n)} \right).$$

Ex. 3: For any $\sigma \in \mathbb{R}$ we define $L^2_\sigma(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \langle x \rangle^\sigma f \in L^2(\mathbb{R}^n)\}$. Let $p \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Prove that $p(x, D)$ (initially defined on $\mathcal{S}(\mathbb{R}^n)$) extends to a bounded operator on $L^2_\sigma(\mathbb{R}^n)$.

Ex. 4: Let $\varphi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, $\varphi(0, 0) = 1$ and $\varphi_\epsilon(x, \xi) := \varphi(\epsilon x, \epsilon \xi)$ for all $x, \xi \in \mathbb{R}^n$ and $\epsilon > 0$. Prove the following statements.

1. $\varphi_\epsilon \in S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for every $\epsilon > 0$.
2. $\{\varphi_\epsilon : 0 < \epsilon < 1\} \subset S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a bounded subset.
3. $\varphi_\epsilon \to 1$ in $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for any $m > 0$.
4. If $u \in \mathcal{S}(\mathbb{R}^n)$, then $\varphi_\epsilon(x, D)u \to u$ as $\epsilon \to 0$ in $\mathcal{S}(\mathbb{R}^n)$.
5. If $u \in L^2(\mathbb{R}^n)$, then $\varphi_\epsilon(x, D)u \to u$ as $\epsilon \to 0$ in $L^2(\mathbb{R}^n)$.
6. If $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, then there exist $c_\epsilon \in S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $c_\epsilon(x, D) = [\varphi_\epsilon(x, D), p(x, D)]$.
7. $\{c_\epsilon : 0 < \epsilon < 1\} \subset S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a bounded subset.

Ex. 5: Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, and define the (unbounded) linear operator $P : L^2(\mathbb{R}^n) \supset \mathrm{Dom}(P) \to L^2(\mathbb{R}^n)$ by

$$Pu := p(x, D)u, \quad u \in \mathrm{Dom}(P) := \mathcal{S}(\mathbb{R}^n).$$

Prove that P is closable and that its closure $\overline{P}$ is given by

$$\overline{P}u = p(x,D)u, \quad u \in \mathrm{Dom}(\overline{P}) = \{u \in L^2(\mathbb{R}^n) : p(x,D)u \in L^2(\mathbb{R}^n)\}.$$

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