Problem Sheet 6

To be discussed on Wednesday, 1.6.2016.

Ex. 1: Determine the symbols of the following operators on $\mathbb{R}^n$: Multiplication by a phase, translation, dilation and rotation. Are any of these in $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for some $m \in \mathbb{R}$?

Ex. 2: (a) Compute the formal adjoint of the partial differential operator $p(x, D) = \sum_{|\alpha|\le m} c_\alpha(x) D^\alpha$.

(b) Let $p \in S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ be the symbol of a ψDO in (x, y)-form such that $p(x, x, \xi) = 0$. Prove that $p(x, D, x) = p_L(x, D)$, where $p_L \in S^{m-1}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$.

Ex. 3: For $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and $0 \le t \le 1$ consider the family of operators $Op_t(a)$, defined, for $u \in \mathcal{S}(\mathbb{R}^n)$, by

$$Op_t(a)u(x) := (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi}\, a(tx + (1-t)y, \xi) u(y) dy d\xi. \qquad (1)$$

(a) Calculate $Op_t(a)$ formally for $a_1(x, \xi) = \xi^\alpha$ and $a_2(x, \xi) = x \cdot \xi$.

(b) Prove the following formula for the formal adjoint: $Op_t(a)^* = Op_{1-t}(\overline{a})$. In particular, conclude that if a is real, then $Op_{1/2}(a)$ is formally self-adjoint.[1]

(c) Let $M \in \mathcal{S}(\mathbb{R}^n)$ and $a(x, \xi) = M(x)$. Prove that $Op_t(a)u(x) = M(x)u(x)$ for $u \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and all $0 \le t \le 1$.

(e) For $a \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ let (1) be defined as an oscillatory integral. Prove that, given $a \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $0 \le t \le 1$, there exists $b \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $Op_0(b) = Op_t(a)$.

Ex. 4: (a) Let Q be a non-singular real symmetric $k \times k$ matrix. For $u \in \mathcal{S}(\mathbb{R}^k)$ define

$$e^{\frac{i}{2}QD\cdot D}\, u(x) := \mathcal{F}^{-1}_{\xi\mapsto x}[e^{\frac{i}{2}Q\xi\cdot\xi}\, \widehat{u}(\xi)](x).$$

[1] $Op_{1/2}(a)$ is called the *Weyl quantization* of the symbol a; it is often written as $a^W(x, D)$.

Prove that

$$e^{\frac{i}{2}QD\cdot D}\,u(x) = \frac{|\det Q|^{-1/2}}{(2\pi)^{k/2}}\,e^{\frac{i\pi}{4}sgn(Q)}\int_{\mathbb{R}^k} e^{-\frac{i}{2}Q^{-1}y\cdot y}\,u(x+y)dy.$$

Here, $sgn(Q)$ denotes the number of positive eigenvalues minus the number of negative eigenvalues of Q.

(b) In particular, for $u \in \mathcal{S}(\mathbb{R}^{2n})$, prove that

$$e^{i\,D_x\cdot D_y}\,u(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\,x'\cdot y'}\,u(x+x',y+y')dx'dy'.$$

(c) Prove that

$$(a\#b)(x,\xi) = e^{i\,D_\xi\cdot D_y}\,(a(x,\xi)b(y,\eta))\,|_{\substack{y=x\\ \eta=\xi}}, \tag{2}$$

$$a^*(x,\xi) = e^{i\,D_x\cdot D_\xi}\,\overline{a}(x,\xi). \tag{3}$$

Jean-Claude Cuenin