Mathematisches Institut
der Universität München — Dr. J.-C. Cuenin

Problem Sheet 5

To be discussed on Wednesday, 25.5.2016.

Ex. 1: (a) Let $p_1(x,\xi) = \sum_{|\alpha|\le m_1} a_\alpha(x)\xi^\alpha$ and $p_2(x,\xi) = \sum_{|\alpha|\le m_2} b_\alpha(x)\xi^\alpha$ with $a_\alpha, b_\alpha \in C_b^\infty(\mathbb{R}^n)$. Compute $p_1(x,D_x)p_2(x,D_x)$ directly as well as with the composition formula.

(b) Let $p_1 \in S^{m_1}_{1,0}(\mathbb{R}^n)$, $p_2 \in S^{m_2}_{1,0}(\mathbb{R}^n)$. Compute the first two terms in the asymptotic expansion of $(p_1 \# p_2)(x,\xi)$.

(c) Let p_1, p_2 be as in (b). Prove that there exists $c \in S^{m_1+m_2-1}_{1,0}(\mathbb{R}^{2n})$ such that

$$p_1(x,D)p_2(x,D) - p_2(x,D)p_1(x,D) = c(x,D).$$

Compute the first term in the asymptotic expansion of c.

Ex. 2: (a) Let $p \in S^m_{1,0}(\mathbb{R}^{2n})$ be an elliptic symbol and let $q \in S^{m-\epsilon}_{1,0}(\mathbb{R}^{2n})$ for some $\epsilon > 0$. Prove that $p + q$ is an elliptic symbol of order m.

(b) Let $p, q, q' \in S^\infty_{1,0}(\mathbb{R}^{2n})$, $r, r' \in S^{-\infty}_{1,0}(\mathbb{R}^{2n})$ be such that

$$p(x,D)q(x,D) = I + r(x,D),$$
$$q'(x,D)p(x,D) = I + r'(x,D).$$

Prove that $q - q' \in S^{-\infty}_{1,0}(\mathbb{R}^{2n})$.

Ex. 3: Consider the partial differential operator of second order

$$p(x,D) := \sum_{i,j=1}^n \partial_{x_i} a_{ij}(x)\partial_j + \sum_{j=1}^n b_j(x)\partial_{x_j} + c(x)$$

with smooth and bounded coefficients. Here, $(a_{ij})_{i,j=1}^n$ is symmetric, and there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2, \quad x,\xi \in \mathbb{R}^n.$$

(a) Determine the symbol of $p(x, D)$ and prove that it is elliptic of order 2.

(b) Explicitly construct a parametrix $q(x, D)$ such that $p(x, D)q(x, D) = I + r(x, D)$, where $r \in S^{-2}_{1,0}(\mathbb{R}^{2n})$.

Ex. 4: Let $(a_j)_{j=0}^{\infty}$ be an arbitrary sequence in $\mathbb{R}$. Show that there exists a function $f \in C^{\infty}(\mathbb{R})$ such that $f^{(j)}(0) = j!a_j$.

Jean-Claude Cuenin