Problem Sheet 4

To be discussed on Wednesday, 11.5.2016.

Ex. 1: Generalization of symbol classes: Let $m \in \mathbb{R}$, $0 \leq \rho, \delta \leq 1$. We say that $a \in S^m_{\rho,\delta}(\mathbb{R}^{2n})$ if $a \in C^\infty(\mathbb{R}^{2n})$ and for any $\alpha, \beta \in \mathbb{N}_0^n$ there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

(a) Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2n})$ and $b \in S^{m'}_{\rho,\delta}(\mathbb{R}^{2n})$. Prove that $\partial_\xi^\alpha \partial_x^\beta a \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}(\mathbb{R}^{2n})$ and that $ab \in S^{m+m'}_{\rho,\delta}(\mathbb{R}^{2n})$.

(b) Which of the following symbols belong to $S^m_{\rho,\delta}(\mathbb{R}^{2n})$?

$$\mathrm{e}^{-t|\xi|^2}, \quad \mathrm{e}^{-\mathrm{i}\, t|\xi|^2}, \quad |\xi|^{-2}, \quad \langle\xi\rangle^{a(x)}, \quad \psi(\xi), \quad \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha, \quad \frac{1-\psi(\xi)}{\mathrm{i}\,\xi_1 + \sum_{j=2}^n |\xi_j|^2}$$

Here, $t > 0$ is a parameter, $a, a_\alpha \in C_b^\infty(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ is such that $\psi(\xi) = 1$ for $|\xi| \leq 1$.

Ex. 2: Generalization of oscillatory integrals: We say that $\phi \in C^\infty(\mathbb{R}^{2n})$ is a *phase function* if

$$\mathrm{Im}(\phi) \geq 0, \quad |\partial^\alpha_{(x,\xi)}\phi(x,\xi)| \leq C_\alpha \quad \text{for all } |\alpha| \geq 2,$$

and there exist $C > 0$, $\rho > 0$ and $R > 0$ such that

$$|\nabla(x,\xi)\phi(x,\xi)| \geq \frac{1}{C}\langle\xi\rangle^\rho, \quad \text{for } |\xi| \geq R.$$

Let $u \in \mathcal{S}(\mathbb{R}^n)$, $a \in S^m_{1,0}(\mathbb{R}^{2n})$, and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ in a neighborhood of $\xi = 0$. Prove that

$$\lim_{\epsilon\to 0} \int\int \mathrm{e}^{\mathrm{i}\,\phi(x,\xi)}\,\chi(\epsilon\xi) a(x,\xi) u(x)\, dx d\xi$$

exists, and estimate its absolute value.

Ex. 3: (a) Let $f \in C^2([-1,1])$, and let $A_j := \sup_{t\in[-1,1]} |f^{(j)}(t)|$ for $j = 0, 2$. Prove that

$$|f'(0)|^2 \leq 4A_0(A_0 + A_2).$$

(b) Prove the following generalization of Corollary 3.10 from the lecture : Let $(a_j)_{j\in\mathbb{N}} \subset \mathcal{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ be a bounded sequence, and let $a \in \mathcal{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$. Assume that $a_j \to a$ as $j \to \infty$ uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^n$. Prove that

$$Os - \int\int \mathrm{e}^{-\mathrm{i}\, y\cdot\eta}\, a_j(y,\eta) dy d\eta \overset{j\to\infty}{\to} Os - \int\int \mathrm{e}^{-\mathrm{i}\, y\cdot\eta}\, a(y,\eta) dy d\eta.$$

Ex. 4: Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ with $\chi(0) = 1$. Let $\chi_\epsilon(x) := \chi(\epsilon x)$. Prove that $\{\chi_\epsilon : 0 < \epsilon < 1\}$ is uniformly bounded in $C^\infty_b(\mathbb{R}^n)$.

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