## Problem Sheet 3

To be discussed on Wednesday, 04.5.2016.

**Ex. 1:** Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $g \in \mathcal{S}(\mathbb{R}^n)$. Prove that

$$\partial_x^\alpha (f * g) = (\partial_x^\alpha f) * g = f * (\partial_x^\alpha g), \quad \mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g] \quad (\text{in } \mathcal{S}'(\mathbb{R}^n))$$

for all $\alpha \in \mathbb{N}_0^n$.

**Ex. 2:** Let $f \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp} f \subseteq \overline{B_R(0)}$. Prove that $\widehat{f}$ is holomorphic in $\mathbb{C}$. Moreover, $|\widehat{f}(\xi)| \leq C\,\mathrm{e}^{R|\operatorname{Im}\xi|}$. Conclude that $\operatorname{supp}\widehat{f}$ cannot be compact unless $\widehat{f} \equiv 0$.

**Ex. 3:** (a) Let $s > n/2$. Prove that $H^s(\mathbb{R}^n) \hookrightarrow C_b^0(\mathbb{R}^n)$ (continuous embedding[1]).

(b) Prove that $\delta_0 \in H^{-s}(\mathbb{R}^n)$ for any $s > n/2$. *Hint:* Use duality.

(c) Let $s > n/2$. Prove that $H^s(\mathbb{R}^n)$ is an algebra under pointwise addition and multiplication.

(d) Let $f \in C^\infty(\mathbb{C})$, $f(0) = 0$, and let $u \in H^s(\mathbb{R}^n)$ with $s > n/2$. Prove that $f \circ u \in H^s(\mathbb{R}^n)$.

**Ex. 4:** Let $p_j \in S_{1,0}^{m_j}(\mathbb{R}^n \times \mathbb{R}^n)$, $m_j \in \mathbb{R}$, $j = 1, 2$, and let $p(x,\xi) = p_1(x,\xi)p_2(x,\xi)$ for all $x, \xi \in \mathbb{R}^n$. Prove that $p \in S_{1,0}^{m_1+m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ and that for every $k \in \mathbb{N}$ there exists $C_k$ (depending only on $k$ and $n$) such that

$$|p|_k^{(m_1+m_1)} \leq C_k |p|_k^{(m_1)} |p|_k^{(m_2)}.$$

**Ex. 5:** Let $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a smooth function which is homogeneous of degree $m \in \mathbb{R}$ for $|\xi| \geq 1$, in the sense that

$$p(x, r\xi) = r^m p(x,\xi) \quad \text{for} \quad |\xi| \geq 1, \quad r \geq 1.$$

[1] i.e. $H^s(\mathbb{R}^n) \subseteq C_b^0(\mathbb{R}^n)$ and $\|f\|_{C_b^0(\mathbb{R}^n)} \leq C_s \|f\|_{H^s(\mathbb{R}^n)}$ for all $f \in C_b^0(\mathbb{R}^n)$. Here we use the notation $H^s(\mathbb{R}^n) := H^{s,2}(\mathbb{R}^n)$.

Moreover, assume that $\partial_\xi^\alpha p(\cdot, \xi) \in C_b^\infty(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$. Prove that $p \in S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$.

**Ex. 6:** (a) Prove that $a \in S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$ if and only if $\langle \xi \rangle^{-m} a \in S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$.

(b) Let $a \in S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$ and $f \in C^\infty(\mathbb{C})$. Prove that $f \circ a \in S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$.

Jean-Claude Cuenin