## Problem Sheet 2

To be discussed on Wednesday, 27.4.2016.

**Ex. 1:** A function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called *homogeneous of degree* $d \in \mathbb{R}$ if $f(rx) = r^d f(x)$ for all $r > 0$ and $x \neq 0$.

(a) Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be continuous and homogeneous of degree $d \in \mathbb{R}$. Prove that there exists a constant $C > 0$ (depending on $f$) such that

$$|f(x)| \leq C|x|^d, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

What is the smallest possible $C$?

(b) Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be $k$-times continuously differentiable and homogeneous of degree $d \in \mathbb{R}$. Prove that $\partial_x^\alpha f$ is homogeneous of degree $d - |\alpha|$ and that

$$|\partial_x^\alpha f(x)| \leq C_\alpha |x|^{d-|\alpha|}, \quad x \in \mathbb{R}^n \setminus \{0\}$$

for all $|\alpha| \leq k$, where $C_\alpha$ depends on $\alpha$ and $f$.

**Ex. 2:** Let $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. Prove that for any $s \in \mathbb{R}$, $\alpha \in \mathbb{N}_0^n$, there exists $C_{s,\alpha} > 0$ such that

$$|\partial_\xi^\alpha \langle \xi \rangle^s| \leq C_{s,\alpha}(1 + |\xi|)^{s-|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

*Hint:* Consider the function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $f(a, x) := (a^2 + |x|^2)^{s/2}$. Use that $f$ is homogeneous of degree $s$.

**Ex. 3:** (a) Prove that

$$|f|''_{k,\mathcal{S}} := \sup_{|\alpha|+|\beta|\leq k} \|x^\alpha D_x^\beta f\|_2, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

defines an equivalent[1] family of semi-norms on $\mathcal{S}(\mathbb{R}^n)$.

(b) Prove that for any $k \in \mathbb{N}$ there is a $C_k > 0$ such that

$$|f|''_{k,\mathcal{S}} \leq C_k \|\langle x \rangle^k \langle D_x \rangle^k f\|_2, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

[1] to the family of seminorms defined in the lecture

(c) Prove that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ there exist $N \in \mathbb{N}_0$ and $C > 0$ such that

$$|\langle f, \varphi \rangle| \leq C \|\langle x \rangle^{2N} \langle D_x \rangle^{2N} \varphi\|_2, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Conclude that $\langle x \rangle^{-2N} \langle D_x \rangle^{-2N} f \in L^2(\mathbb{R}^n)$. *Hint:* Use the Riesz representation theorem.

**Ex. 4:** Prove that for every $f \in \mathcal{S}'(\mathbb{R}^n)$ there exists a sequence $(f_k)_k \subset C_0^\infty(\mathbb{R}^n)$ such that $\lim_{k\to\infty} f_k = f$ in $\mathcal{S}'(\mathbb{R}^n)$.

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