## Problem Sheet 10

To be discussed on Wednesday, 29.6.2016.

**Ex. 1:** Let $P = (x_2 D_{x_2})^2 - D_{x_1}^2 + 2\,\mathrm{i}\,\mu x_2 D_{x_2}$, $\mu \geq 0$.

(a) Show that for every $\beta \in \mathbb{R}$ the distributions $u_\beta(x) = H(x_2) x_2^{\alpha(\beta)}\,\mathrm{e}^{\beta x_1}$ with $\alpha(\beta) = \mu + (\mu^2 + \beta^2)^{1/2}$ are in $\mathcal{D}'(\mathbb{R}^2)$ and satisfy $Pu_\beta = 0$. (Here $H$ is the Heaviside function.)

(b) Show that $WF(u_\beta) \subset \{(x_1, x_2; \xi_1, \xi_2) : x_2 = \xi_1 = 0, \xi_2 \neq 0\}$.

(c) Show that the inclusion is in fact an equality.

**Ex. 2:** Denote by $\mathbf{1}_A$ the characteristic function of a subset $A$ of $\mathbb{R}^n$. Determine $WF(u)$ if

(a) $u = \mathbf{1}_A \in \mathcal{D}'(\mathbb{R}^n)$ with $A = \{x \in \mathbb{R}^n : \varphi(x) > 0\}$, $\varphi \in C^\infty(\mathbb{R}^n; \mathbb{R})$, $\nabla\varphi(x) \neq 0$ if $\varphi(x) = 0$.

(b) $u = \mathbf{1}_{A_1} - a\mathbf{1}_{A_2} \in \mathcal{D}'(\mathbb{R}^2)$, $a \in \mathbb{R}$, where $A_1 = \{x : x_1 > 0, x_2 > 0\}$, $A_2 = \{x : x_1 < 0, x_2 < 0\}$.

**Ex. 3:** Let $u \in \mathcal{D}'(\mathbb{R}^4)$. Assume that $u$ is $C^\infty$ in a neighborhood of the axis $\{x \in \mathbb{R}^4 : x_1 = x_2 = x_3 = 0\}$. Assume also that $D_{x_1} u \in C^\infty(\mathbb{R}^4)$ and $(D_{x_2} - x_1 D_{x_3})u \in C^\infty(\mathbb{R}^4)$. Show that $u \in C^\infty(\mathbb{R}^4)$.

**Ex. 4:** Let $P$ be the differential operator on $\mathbb{R}^2$ given by

$$P = (x_2 D_{x_2}^2) - D_{x_1}^2 + f(x) D_{x_1} + g(x) D_{x_2} + h(x)$$

with $f, g, h \in C^\infty(\mathbb{R}^2)$.

(a) Determine the bicharacteristic strip of $P$ passing through the point $(x_1, x_2; \xi_1, \xi_2) = (0, 1; 1, 1)$ up to reparametrization. (Note that the principal symbol can be factorized.)

(b) Determine all the bicharacteristics passing above $x = (0, 1)$ and their projections onto $\mathbb{R}^2_x$.

(c) Consider the particular case $f = g = h = 0$. Show that if $Pu \in C^\infty(\mathbb{R}^2)$ and $u$ is $C^\infty$ near $(0, 0)$, then $u$ is $C^\infty$ near $\{x : x_2 = 0\}$.

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