Problem Sheet 1

To be discussed on Wednesday, 20.4.2016.

Ex. 1: Prove Theorem 2.1 from the lecture.

Ex. 2: (a) Let $f \in C^k(\mathbb{R}^n)$ be such that $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for $|\alpha| \leq k$. Prove that $|\widehat{f}(\xi)| \leq C(1+|\xi|)^{-k}$ for some constant C (depending on f).

(b) Assume that $(1+|\cdot|)^k f \in L^1(\mathbb{R}^n)$. Prove that $\widehat{f} \in C^k(\mathbb{R}^n)$.

Ex. 3: Let $f, g \in L^1(\mathbb{R}^n)$. Prove that $\mathcal{F}[f * g](\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$ for all $\xi \in \mathbb{R}^n$.

Ex. 4: (a) Prove that $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

(b) Prove that $\mathcal{S}(\mathbb{R}^n)$, together with the family of semi-norms $|\cdot|_{m,\mathcal{S}}$, $m \in \mathbb{N}$, defined by

$$|f|_{m,\mathcal{S}} := \sup_{|\alpha|+|\beta| \leq m} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)|, \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{1}$$

is a Fréchet space. (*Hint:* You may use without proof that $C_b^k(\mathbb{R}^n)$ is complete for any $k \in \mathbb{N}_0$.)

(c) Show that the semi-norms $|\cdot|'_{\mathcal{S},m}$, $m \in \mathbb{N}$, defined by

$$|f|'_{\mathcal{S},m} := \sup_{k+|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} (1+|x|)^k |\partial_x^\alpha f(x)|, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

are equivalent to the semi-norms (1). More precisely, prove that for every $m \in \mathbb{N}$ there exist constants C_m, c_m such that

$$c_m |f|_{\mathcal{S},m} \leq |f|'_{\mathcal{S},m} \leq C_m |f|_{\mathcal{S},m}.$$

Ex. 5: (a) Prove that for every $\alpha \in \mathbb{N}_0^n$ and $m \in \mathbb{N}$ there are constants $C_{m,\alpha}, C_{m,\alpha'} > 0$ such that

$$|x^\alpha f|_{\mathcal{S},m} \leq C_{m,\alpha} |f|_{\mathcal{S},m+|\alpha|}, \quad |\partial_x^\alpha f|_{\mathcal{S},m} \leq C'_{m,\alpha} |f|_{\mathcal{S},m+|\alpha|}$$

uniformly in $f \in \mathcal{S}(\mathbb{R}^n)$. Conclude that for every $\alpha, \beta \in \mathbb{N}_0^n$ and $m \in \mathbb{N}$, there is a constants $C_{m,\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha (x^\beta f)| \leq C_{m,\alpha,\beta} |f|_{\mathcal{S},m+|\alpha|+|\beta|}$$

uniformly in $f \in \mathcal{S}(\mathbb{R}^n)$.

(b) Recall that $g \in C^\infty_{\text{poly}}(\mathbb{R}^n)$ if and only if for each $\alpha \in \mathbb{N}_0^n$ there exist $C_\alpha > 0$ and $m(\alpha) \in \mathbb{N}_0$ such that

$$|\partial_x^\alpha g(x)| \leq C_\alpha (1 + |x|)^{m(\alpha)}, \quad x \in \mathbb{R}^n. \tag{2}$$

Let $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in C^\infty_{\text{poly}}(\mathbb{R}^n)$. Prove that $fg \in \mathcal{S}(\mathbb{R}^n)$, with

$$|fg|_{\mathcal{S},m} \leq C_{m,g} |f|_{\mathcal{S},m+\max_{|\delta|\leq m} m(\delta)}, \quad m \in \mathbb{N}_0^n,$$

where the constant $C_{m,g}$ depends on m and g only. Conclude that the multiplication operator $M_g : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $(M_g f)(x) := g(x)f(x)$, is bounded.

(c) Prove that the differential operator

$$Lu(x) = \sum_{|\alpha|\leq m} c_\alpha(x) D_x^\alpha u(x), \quad u \in \mathcal{S}(\mathbb{R}^n),$$

with $c_\alpha \in C^\infty_{\text{poly}}(\mathbb{R}^n)$, defines a bounded linear operator $L : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.

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