Chapter 6

Phase transitions in quantum spin systems

Let $\mathcal{A} = \overline{\bigcup_{\Lambda\in\mathcal{F}(\Gamma)} \mathcal{A}_\Lambda}$ be the C*-algebra of a quantum spin system.

Definition 17. *An* interaction *on $\mathcal{A}$ is a map defined on $\mathcal{F}(\Gamma)$ such that for $X \in \mathcal{F}(\Gamma)$, $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$. Furthermore, for any non-negative function $\xi : \mathcal{F}(\Gamma) \to [0,\infty)$,*

$$\mathcal{B}_\xi := \left\{ \Phi : \|\Phi\|_\xi := \sup_{x\in\Gamma} \sum_{X\ni x} \|\Phi(X)\|\xi(X) < \infty \right\}.$$

*is a Banach space of interactions. Finally, an N-*body interaction *is defined by the condition $\Phi(X) = 0$ if $|X| \neq N$.*

In the case of a N body interaction, one writes $\Phi(x_1,\dots,x_N), x_i \in \Gamma$. A simple example is $\xi(X) = 1$ implying an integrable decay. We shall use the following: Let D be the maximal degree in Γ and $\mathrm{diam}(X) := \max\{d(x,y) : x,y \in X\}$ for any $X \in \mathcal{F}(\Gamma)$. For any $\lambda > 0$, denote

$$\mathcal{B}_\lambda := \mathcal{B}_{\xi_\lambda}, \qquad \xi_\lambda(X) := |X| D^{2|X|} \mathrm{e}^{\lambda \mathrm{diam}(X)}.$$

Now, for $\Lambda \in \mathcal{F}(\Gamma)$, the Hamiltonian is the sum of interactions within Λ, namely

$$H_\Lambda := \sum_{X\subset\Lambda} \Phi(X)$$

and for $A \in \mathcal{A}$,

$$\tau_t^{\Phi,\Lambda}(A) = \mathrm{e}^{\mathrm{i}tH_\Lambda} A \mathrm{e}^{-\mathrm{i}tH_\Lambda}$$

is a strongly continuous one parameter group of *-automorphisms of $\mathcal{A}$. Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}(\Gamma)$ such that $\Lambda_n \subset \Lambda_m$ is $n \leq m$ and for any $x \in \Gamma$ there exists n_0 such that $x \in \Lambda_n$ for all $n \geq n_0$.

Theorem 40. *Let $\lambda > 0$ and $\Phi \in \mathcal{B}_\lambda$. There exists a strongly continuous one parameter group of *-automorphisms $\{\tau_t^\Phi : t \in \mathbb{R}\}$ of $\mathcal{A}$ such that, for any $A \in \mathcal{A}$,*

$$\lim_{n\to\infty} \|\tau_t^{\Phi,\Lambda_n}(A) - \tau_t^\Phi(A)\| = 0,$$

for all $t \in \mathbb{R}$. The convergence is uniform for t in a compact set and the limit is independent of the sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$.

The theorem is an immediate consequence of the *Lieb-Robinson bound*: For $\Phi \in \mathcal{B}_\lambda$ there is $v_\lambda > 0$, such that for any $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, and $X \cup Y \in \Lambda$,

$$\|[\tau_t^{\Phi,\Lambda}(A), B]\| \leq C\|A\|\|B\| \min\{|X|, |Y|\} \exp(-\lambda(d(X,Y) - v_\lambda|t|))$$

where the constant C is independent of Λ. This is a propagation estimate: up to exponentially small corrections, the support of A grows linearly with time, with velocity v_λ. Since, for $n \geq m$ with $A \in \mathcal{A}_\Lambda, \Lambda \subset \Lambda_m$,

$$\tau_t^{\Phi,\Lambda_n}(A) - \tau_t^{\Phi,\Lambda_m}(A) = \int_0^t \frac{d}{ds}\left(\tau_s^{\Phi,\Lambda_n} \circ \tau_{t-s}^{\Phi,\Lambda_m}(A)\right) ds = \int_0^t \tau_s^{\Phi,\Lambda_n}(\delta^{\Phi,\Lambda_n} - \delta^{\Phi,\Lambda_m})\tau_{t-s}^{\Phi,\Lambda_m}(A) ds$$

and $\delta^{\Phi,\Lambda_n} - \delta^{\Phi,\Lambda_m} = \sum_{X: X\cap(\Lambda_n\setminus\Lambda_m)\neq\emptyset}[\Phi(X), \cdot]$, we have

$$\begin{aligned}
\|\tau_t^{\Phi,\Lambda_n}(A) - \tau_t^{\Phi,\Lambda_m}(A)\| &\leq \int_0^t \sum_{x\in\Lambda_n\setminus\Lambda_m} \sum_{X\ni x} \|[\Phi(X), \tau_{t-s}^{\Phi,\Lambda_m}(A)]\| ds \\
&\leq C|\Lambda|\|A\| \sum_{x\in\Lambda_n\setminus\Lambda_m} \sum_{X\ni x} \|\Phi(X)\| \exp(-\lambda(d(X,\Lambda) - v_\lambda|t|) \\
&\leq \tilde{C}|\Lambda|\|A\|\|\Phi\|_\lambda \exp(-\lambda d(\Gamma \setminus \Lambda_m, \Lambda)) \exp(\lambda v_\lambda|t|).
\end{aligned}$$

This vanishes uniformly as $m \to \infty$ for t in a compact interval, and $\{\tau_t^{\Phi,\Lambda_n}(A)\}_{n\in\mathbb{N}}$ is Cauchy.

A typical example is the Heisenberg models. Here $\Gamma = \mathbb{Z}^d$, and $\mathcal{H}_x = \mathbb{C}^{2s+1}$ is the representation space of $SU(2)$ with generator S^1, S^2, S^3. The Heisenberg Hamiltonian is given by

$$H_{\Lambda,J,h} = \sum_{\{x,y\}\in\Lambda\times\Lambda} \sum_{i=1}^3 J_{xy}^i S_x^i S_y^i - h\sum_{x\in\Lambda} S_x^3, \quad h > 0,$$

with some decay on $|J_{xy}^i|$ in $d(x,y)$. This defines a translation invariant Hamiltonian if $J_{xy}^i = J^i$ for all $\{x,y\} \in \Gamma \times \Gamma$ and an $SU(2)$-invariant interaction if $J_{xy}^i = J_{xy}$ for $i = 1,2,3$.

6.1 The theorem of Mermin & Wagner

We now apply Theorem 39 to the concrete case of low dimensional quantum spin systems and obtain a general form of the theorem of Mermin and Wagner. Note that this only one version of the theorem, namely about the absence of symmetry breaking, which does not necessarily exclude other types of phase transitions. The original proof in the generality given here is due to Fröhlich-Pfister. For simplicity, we consider $\mathcal{H}_x = \mathcal{H}$ for all $x \in \Gamma$.

Let G be a compact connected Lie group and let $G \ni g \mapsto U_g$ be a strongly continuous unitary representation of G on $\mathcal{H}$. This induces a group of *-automorphisms of $\mathcal{A}_{\{x\}}$ by $\alpha_g^{\{x\}}(A) = U_g^* A U_g$, and the tensor product representation $\otimes_{x\in\Lambda} U_g$ induces the tensor action α_g^Λ on $\mathcal{A}_\Lambda$, for any $\Lambda \in \mathcal{F}(\Gamma)$. Hence, this defines a strongly continuous group of *-automorphisms on $\mathcal{A}_{\mathrm{loc}}$ which extends by continuity to $\{\alpha_g : g \in G\}$ on $\mathcal{A}$. Note that the complete system is rotated by the same element g, a 'global gauge transformation'. A typical example is $\mathcal{H} = \mathbb{C}^{2s+1}$ carrying the spin-s representation of $G = SU(2)$, namely $U_g = \exp(2\pi i g \cdot S)$, where g is an element of the unit ball and S is the vector of spin matrices.

Theorem 41. *Let $\mathcal{A}$ be as above with $\Gamma = \mathbb{Z}^2$, $\{\alpha_g : g \in G\}$ the action of the compact connected Lie group G, and Φ a G-invariant two-body interaction, namely*

$$\alpha_g(\Phi(x,y)) = \Phi(x,y), \quad \textit{for all } x, y \in \mathbb{Z}^2, g \in G.$$

If

$$\sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}\|\Phi(x,y)\|d(x,y)^2<\infty$$

then for any $0<\beta<\infty$ *and any* (τ^Φ,β)*-KMS state* ω*,*

$$\omega\circ\alpha_g=\omega,\quad \textit{for all } g\in G.$$

For a translation and rotation invariant interaction, the sharp condition is $\|\Phi(x,y)\|\le Cd(x,y)^{-4}$: there are models with an interaction decaying as $d(x,y)^{-4+\epsilon}$ for which phase transitions are known to occur.

Proof. We consider a one dimensional subgroup H of G, and since G is compact, $H\simeq\mathbb{R}/\mathbb{Z}=\mathbb{S}^1$. We consider the generator $S=S^*\in\mathcal{L}(\mathcal{H})$, namely $U_\phi=\exp(\mathrm{i}\phi S)$ for $\phi\in[0,2\pi)$.

For $m\in\mathbb{N}$, let $\Lambda_m=[-m,m]\cap\mathbb{Z}^2$. Let ϕ be fixed, and let $\varphi_m:\mathbb{Z}^2\to[0,2\pi)$ be given by

$$\varphi_m(x)=\begin{cases}\phi & x\in\Lambda_m\\ \phi(2-\max\{|x_1|,|x_2|\}/m) & x=(x_1,x_2)\in\Lambda_{2m}\setminus\Lambda_m\\ 0 & \text{otherwise}\end{cases}$$

and finally

$$U_\phi(m):=\bigotimes_{x\in\Lambda_{2m}}U_x(\varphi_m(x))\in\mathcal{A}_{\Lambda_{2m}}\subset D(\delta),$$

which slowly interpolates between a full rotation on Λ_m and no rotation outside of Λ_{2m}.

Let $A\in\mathcal{A}_{\mathrm{loc}}$ and $m_0:=\min\{m\in\mathbb{N}:A\in\mathcal{A}_{\Lambda_m}\}$. We have $U_\phi(m)^*AU_\phi(m)=\alpha_\phi(A)$ for all $m\ge m_0$, so that Assumption (A) of Theorem 39 holds. We now claim that Assumption (Bii) also holds. Noting that for $A\in\mathcal{A}_\Lambda$, $\delta(A)=\mathrm{i}\sum_{\{x,y\}\cap\Lambda\neq\emptyset}[\Phi(x,y),A]$, we compute

$$U_\phi(m)^*\delta(U_\phi(m))=\mathrm{i}\sum_{\{x,y\}\in\mathbb{Z}^2\setminus\mathcal{N}_m}U_\phi(m)^*\Phi(x,y)U_\phi(m)-\Phi(x,y)$$

where $\mathcal{N}_m=\{\{x,y\}:x,y\in\Lambda_m\text{ or }x,y\in\mathbb{Z}^2\setminus\Lambda_{2m}\}$ by the symmetry of the interaction and the support of $U_m(\phi)$. Denote $U_\phi(m)^*\delta(U_\phi(m))+U_\phi(m)\delta(U_\phi(m)^*)=\mathrm{i}\sum_{x,y}\Delta_m(x,y)$. Note that

$$\varphi_m(x)S_x+\varphi_m(y)S_y=\frac{\varphi_m(x)+\varphi_m(y)}{2}(S_x+S_y)+\frac{\varphi_m(x)-\varphi_m(y)}{2}(S_x-S_y):=E_m(x,y)+O_m(x,y)$$

with $[E_m(x,y),O_m(x,y)]=0$ since $[S_x,S_y]=0$. Since, moreover, $E_m(x,y)$ generates the same rotation by $(\varphi_m(x)+\varphi_m(y))/2$ at both x and y, and by the symmetry of the interaction,

$$U_\phi(m)^*\Phi(x,y)U_\phi(m)=\mathrm{e}^{-\mathrm{i}O_m(x,y)}\mathrm{e}^{-\mathrm{i}E_m(x,y)}\Phi(x,y)\mathrm{e}^{\mathrm{i}E_m(x,y)}\mathrm{e}^{\mathrm{i}O_m(x,y)}=\mathrm{e}^{-\mathrm{i}O_m(x,y)}\Phi(x,y)\mathrm{e}^{\mathrm{i}O_m(x,y)}.$$

which has the commutator expansion

$$U_\phi(m)^*\Phi(x,y)U_\phi(m)-\Phi(x,y)=\sum_{k\ge1}\frac{i^k}{k!}\mathrm{ad}^k_{O_m(x,y)}(\Phi(x,y)).$$

Noting that $U_\phi(m)\delta(U_\phi(m)^*)=U_{-\phi}(m)^*\delta(U_{-\phi}(m))$ and $O_m(x,y)$ is odd under $\phi\to-\phi$, all odd terms in the series of $\Delta_m(x,y)$ cancel, yielding the estimate

$$\|\Delta_m(x,y)\|\le2\sum_{k\ge1}\frac{1}{(2k)!}\frac{1}{2^{2k}}|\varphi_m(x)-\varphi_m(y)|^{2k}\|\mathrm{ad}^{2k}_{S_x-S_y}(\Phi(x,y))\|.$$

It remains to observe that $|\varphi_m(x) - \varphi_m(y)| \leq |\phi| \min\{1, d(x,y)/m\}$, so that

$$|\varphi_m(x) - \varphi_m(y)|^{2k} \leq |\phi|^{2k} \left(\frac{d(x,y)}{m}\right)^2$$

and to carry out the spatial sum to obtain

$$\sum_{\{x,y\} \in \mathbb{Z}^2 \setminus \mathcal{N}_m} \|\Delta_m(x,y)\| \leq \frac{4\mathrm{e}^{2\|S\||\phi|}}{m^2} \sum_{x \in \Lambda_{2m}} \sum_{y \in \mathbb{Z}^2} \|\Phi(x,y)\| d(x,y)^2 =: M < \infty$$

which is finite by assumption after estimating the sum by $(2m+1)^2 \sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} (\cdots)$.

For any $n \in \mathbb{N}$, we claim that $\omega \circ \alpha_{\pi/2^n} = \omega$. This follows from a recursive application of Theorem 39 starting with the observation that $\alpha_\pi^2 = \mathrm{id}$. Finally, the set $D := \{\phi \in \mathbb{S}^1 : \phi = \sum_{n=0}^N a_n(\pi/2^n), a_n \in \mathbb{Z}, N \in \mathbb{N}\}$ is dense in $\mathbb{S}^1$. For any $A \in \mathcal{A}$, the function $\phi \mapsto \xi_A(\phi) := \omega(\alpha_\phi(A) - A)$ is continuous and $\xi_A(\phi) = 0$ if $\phi \in D$. Hence $\xi_A(\phi) = 0$ for all $\phi \in \mathbb{S}^1$. $\square$

Possible extensions following the same ideas with adapted assumptions include one-dimensional models, short range N-body interactions, and non-translation invariant models with possibly different representations of G at different points of Γ.

6.2 Existence of a phase transition in the Heisenberg model

In this section, we shall prove the existence of a phase transition at positive temperature for the antiferromagnetic Heisenberg model, following the original proof of Dyson-Lieb-Simon (1978). The proof relies on a spectral property of the Hamiltonian, *reflection positivity* which fails for the ferromagnetic model. Although a proof of phase transition in that case is still an open problem, recent progress has been made by Corregi, Giuliani and Seiringer (2013), who compute the free energy at low temperature.

We consider $\Lambda := \{-L/2, \cdots, L/2\}^d, L \in 2\mathbb{N}$ understood with periodic boundary conditions, and let E_Λ be the set of nearest neighbour pairs. The translation invariant, spin-S Heisenberg Hamiltonian is written as

$$H_\Lambda^{(u)} := -2 \sum_{\{x,y\} \in E_\Lambda} \left(S_x^1 S_y^1 + u S_x^2 S_y^2 + S_x^3 S_y^3\right), \qquad u \in [-1,1].$$

The case $u = 1$ is the ferromagnet, $u = 0$ the 'XY model', while $u = -1$ locally unitarily equivalent to the antiferromagnet on a bipartite lattice. Indeed, assume that $\Lambda = \Lambda_A \cup \Lambda_B$, with $\{x,y\} \in E_\Lambda$ implies $x \in \Lambda_A, y \in \Lambda_B$ or $x \in \Lambda_B, y \in \Lambda_A$ and note that local rotations by π along the 2 axis, generated by S_x^2, yield $\exp(-\mathrm{i}\pi S_x^2) S_x^j \exp(\mathrm{i}\pi S_x^2) = (-1)^j S_x^j$. It follows that conjugation $U_\Lambda^* H_\Lambda^{(-1)} U_\Lambda$ with the unitary $U_\Lambda := \prod_{x \in \Lambda_A} \exp(\mathrm{i}\pi S_x^2)$ yields the antiferromagnet.

Given the Gibbs state $\omega_{\beta,\Lambda}^{(u)}$, we are interested proving the existence of *long-range order*

$$\lim_{|x| \to \infty} \liminf_{\Lambda \to \mathbb{Z}^d} \omega_{\beta,\Lambda}^{(u)}(S_0^3 S_x^3) > 0, \qquad d \geq 3, \tag{6.1}$$

for β sufficiently large[1]. This implies for the the magnetisation $M_\Lambda := |\Lambda|^{-1} \sum_{x \in \Lambda} S_x^3$ that

$$\liminf_{\Lambda \to \mathbb{Z}^d} \omega_{\beta,\Lambda}^{(u)}(M_\Lambda^2) > 0, \qquad d \geq 3, \quad \beta \text{ sufficiently large}$$

[1]In the case of the antiferromagnet,

$$(-1)^{d(0,x)} \omega_{\beta,\Lambda}^{\mathrm{antiferro}}(S_0^3 S_x^3) = (-1)^{d(0,x)} \omega_{\beta,\Lambda}^{(-1)}(U_\Lambda^* S_0^3 S_x^3 U_\Lambda) = \omega_{\beta,\Lambda}^{(-1)}(S_0^3 S_x^3)$$

which remains uniformly bounded away from 0. This is called 'Néel ordering'.

which corresponds to the intuition of macroscopic fluctuations in the bulk magnetisation and the presence of multiple phases. In fact, a little more abstract nonsense would show that long-range order is inconsistent with the invariance of any extremal KMS state.

Theorem 42. *Consider $H_\Lambda^{(u)}$ for $u \in [-1, 0]$ and spin S. Then,*

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\omega^{(u)}_{\beta,\Lambda}(S^3_0 S^3_x) \geq \frac{1}{3}S(S+1) - \frac{1}{\sqrt{2}|\Lambda|}\sum_{k\in\Lambda^*\setminus\{0\}}\sqrt{\frac{E^{(u)}(k)}{E(k)}} - \frac{1}{2\beta|\Lambda|}\sum_{k\in\Lambda^*\setminus\{0\}}\frac{1}{E(k)} \tag{6.2}$$

where Λ^* is the lattice dual to Λ and

$$E(k) := 2\sum_{i=1}^d (1-\cos(k_i)), \quad E^{(u)}(k) := \sum_{i=1}^d (1-u\cos(k_i))\omega^{(u)}_{\beta,\Lambda}(S^1_0 S^1_{e_i}) + (u-\cos(k_i))\omega^{(u)}_{\beta,\Lambda}(S^2_0 S^2_{e_i}).$$

Note that for any state ν, $|\nu(S^j_x S^j_y)| \leq \nu((S^j_x)^2)^{1/2}\nu((S^j_y)^2)^{1/2} \leq \nu(\vec{S}^2) = S(S+1)$ so that $|E^{(u)}(k)| \leq 4dS(S+1)$. Furthermore, $1-\cos(x) = (1/2)x^2 + O(x^4)$ as $x \to 0$ so that $E(k)^{-1}$ is integrable if $d \geq 3$, and $E(k)^{-1/2}$ is integrable if $d \geq 2$. Hence, if $d \geq 3$, there exist $0 < C_d, \kappa_d < \infty$ such that

$$\liminf_{\Lambda\to\mathbb{Z}^d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\omega^{(u)}_{\beta,\Lambda}(S^3_0 S^3_x) \geq \frac{1}{3}S(S+1) - \kappa_d\sqrt{S(S+1)} - \frac{C_d}{\beta}$$

and the lower bound is strictly positive for S large enough and all $\beta \geq \beta_c = \beta_c(d, S)$. In turn, this implies long-range order, (6.1). Note that improved estimates allow to extend the statement to $d \geq 3$ and all $S \in (1/2)\mathbb{N}$.

Let $v : \mathbb{Z}^d \to \mathbb{R}$ and $h := \Delta v$, namely $h_x := \sum_{y:\{x,y\}\in E_\Lambda}(v_y - v_x)$. In $l^2(\Lambda)$,

$$\langle f, -\Delta g\rangle = \sum_{\{x,y\}\in E_\Lambda}(f_y - f_x)(g_y - g_x),$$

and in particular $\langle v, -\Delta v\rangle = \|h\|^2$.

Let

$$H^{(u)}_\Lambda(v) := H^{(u)}_\Lambda - \sum_{x\in\Lambda} h_x S^3_x,$$

to which we associate the partition function $Z^{(u)}_{\beta,\Lambda}(v) = \mathrm{Tr}\left(\exp(-\beta H^{(u)}_\Lambda(v))\right)$ and

$$\tilde{Z}^{(u)}_{\beta,\Lambda}(v) := Z^{(u)}_{\beta,\Lambda}(v)\mathrm{e}^{-\frac{1}{4}\beta\langle v,-\Delta v\rangle}$$

Let R be a reflection map of Λ and let $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_2 = R\Lambda_1$. Furthermore, $v_1 := v\restriction_{\Lambda_1}$, $v_2 := v\restriction_{\Lambda_2}$ and we shall write $v = v_1|v_2$.

We now exhibit the full structure of the proof.

Lemma 43. *If $u \leq 0$, then for any reflection R,*

$$\tilde{Z}^{(u)}_{\beta,\Lambda}(v_1|v_2)^2 \leq \tilde{Z}^{(u)}_{\beta,\Lambda}(v_1|Rv_1)\tilde{Z}^{(u)}_{\beta,\Lambda}(Rv_2|v_2)$$

Lemma 44. *If $u \leq 0$,*

$$Z^{(u)}_{\beta,\Lambda}(v) \leq Z^{(u)}_{\beta,\Lambda}(0)\mathrm{e}^{\frac{1}{4}\beta\langle v,-\Delta v\rangle}$$

Lemma 45. *If $u \leq 0$, and for any $k \in \Lambda^* \setminus \{0\}$,*

$$\widehat{(S_0^3, S_\cdot^3)}_\beta^{(u)}(k) \leq (2\beta E(k))^{-1}$$

where $(\cdot,\cdot)_\beta$ denotes Duhamel's two-point function,

$$(A,B)_\beta^{(u)} := \frac{1}{Z_{\beta,\Lambda}^{(u)}} \int_0^1 \mathrm{Tr}\left(\mathrm{e}^{-\beta s H_\Lambda^{(u)}} A \mathrm{e}^{-\beta(1-s)H_\Lambda^{(u)}} B\right) ds$$

Lemma 46. *For any $k \in \Lambda^* \setminus \{0\}$ such that $\widehat{(S_0^3, S_x^3)}_\beta^{(u)}(k) \leq (2\beta E(k))^{-1}$,*

$$\widehat{\omega_{\beta,\Lambda}^{(u)}(S_0^3 S_\cdot^3)}(k) \leq \sqrt{\frac{E^{(u)}(k)}{2E(k)}} + \frac{1}{2\beta E(k)}.$$

Proof of Theorem 42. Let $C_\Lambda(x) := \omega_{\beta,\Lambda}^{(u)}(S_0^3 S_x^3)$. We have

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda} \omega_{\beta,\Lambda}^{(u)}(S_0^3 S_x^3) = \widehat{C_\Lambda}(0) = C_\Lambda(0) - \sum_{k\in\Lambda^*\setminus\{0\}} \widehat{C_\Lambda}(k).$$

Furthermore, we note that $C_\Lambda(0) = \omega_{\beta,\Lambda}^{(u)}((S_0^3)^2) = (1/3)\omega_{\beta,\Lambda}^{(u)}(\vec{S}^2) = (1/3)S(S+1)$, which concludes the proof with Lemma 46. □

We should remark that in finite volume, the Gibbs state has all symmetries of the Hamiltonian since its density matrix is a function of the Hamitonian. In particular, $\omega_{\beta,\Lambda}^{(u)}(S_0^3) = 0$ or $\omega_{\beta,\Lambda}^{(u)}(M_\Lambda) = 0$ and their respective limits likewise. The limiting state must be a non-trivial superposition of extremal KMS states which break the $SU(2)$ symmetry. Here, we consider $m_{\mathrm{sp}} := \liminf_{\Lambda\to\mathbb{Z}^d} \omega_{\beta,\Lambda}^{(u)}(|M_\Lambda|)$, namely the spontaneous magnetisation. One could also add a 'transverse magnetic field' to the Hamiltonian, namely $h\sum_{x\in\Lambda} S_x^3$ and study either the residual magnetisation, $m_{\mathrm{res}} := \lim_{h\to 0^+} \liminf_{\Lambda\to\mathbb{Z}^d} \omega_{\beta,h,\Lambda}^{(u)}(M_\Lambda)$, namely whether the system 'remembers' an external magnetic field which breaks the symmetry. It turns out that $m_{\mathrm{res}} \geq m_{\mathrm{sp}}$ and $m_{\mathrm{sp}} = 0$ if and only if $\liminf_{\Lambda\to\mathbb{Z}^d} \omega_{\beta,\Lambda}^{(u)}(M_\Lambda^2) = 0$, see exercises.

Proof of Lemma 43. Let $\mathcal{H} = \mathcal{K}\otimes\mathcal{K}$, with $\dim\mathcal{K} < \infty$, and let $A, B, C_1, \dots C_l, D_1, \dots D_l \in \mathcal{L}(\mathcal{K})$ be real matrices and $h_1, \dots h_l \in \mathbb{R}$. Then,

$$\begin{aligned}\mathrm{Tr}&\left[\mathrm{e}^{A\otimes 1 + 1\otimes B - \sum_{k=1}^l (C_k\otimes 1 - 1\otimes D_k - h_k)^2}\right]^2 \\ &\leq \mathrm{Tr}\left[\mathrm{e}^{A\otimes 1 + 1\otimes A - \sum_{k=1}^l (C_k\otimes 1 - 1\otimes C_k)^2}\right] \mathrm{Tr}\left[\mathrm{e}^{B\otimes 1 + 1\otimes B - \sum_{k=1}^l (D_k\otimes 1 - 1\otimes D_k)^2}\right]\end{aligned} \tag{6.3}$$

Indeed (in the case $l = 1$), we first apply Trotter's product formula

$$\mathrm{e}^{A\otimes 1 + 1\otimes B - (C\otimes 1 - 1\otimes D - h)^2} = \lim_{n\to\infty}\left(\mathrm{e}^{\frac{1}{n}A\otimes 1}\mathrm{e}^{\frac{1}{n}1\otimes B}\mathrm{e}^{-\frac{1}{n}(C\otimes 1 - 1\otimes D - h)^2}\right)^n$$

and the operator identity

$$\mathrm{e}^{-M^2} = (4\pi)^{-1/2}\int_{\mathbb{R}} \mathrm{e}^{-s^2/4}\mathrm{e}^{\mathrm{i}sM} ds$$

to write the trace as

$$(4\pi)^{-n/2}\int ds_1\cdots ds_n \mathrm{Tr}\left[\left(\mathrm{e}^{\frac{1}{n}A\otimes 1}\mathrm{e}^{\mathrm{i}\frac{s_1}{\sqrt{n}}C\otimes 1}\cdots \mathrm{e}^{\frac{1}{n}A\otimes 1}\mathrm{e}^{\mathrm{i}\frac{s_n}{\sqrt{n}}C\otimes 1}\right)\right]$$
$$\overline{\mathrm{Tr}\left[\left(\mathrm{e}^{\frac{1}{n}1\otimes B}\mathrm{e}^{\mathrm{i}\frac{s_1}{\sqrt{n}}1\otimes D}\cdots \mathrm{e}^{\frac{1}{n}1\otimes B}\mathrm{e}^{\mathrm{i}\frac{s_n}{\sqrt{n}}1\otimes D}\right)\right]}\mathrm{e}^{\mathrm{i}\frac{h\sum_{i=1}^n s_i}{\sqrt{n}}}\mathrm{e}^{-\frac{\sum_{i=1}^n s_i^2}{4}}$$

where we noted that matrices acting on different factors commute, that $\mathrm{Tr}(M\otimes 1)(1\otimes N) = \mathrm{Tr}(M\otimes 1)\mathrm{Tr}(1\otimes N)$, and the reality of the matrices to take the complex conjugate (not the adjoint) without reversing the order of the matrices. Cauchy-Schwarz's inequality for the s-integrals now yields

$$|\cdot|^2 \le \frac{1}{(4\pi)^{n/2}}\int \mathrm{Tr}\prod_{i=1}^n \mathrm{e}^{\frac{1}{n}A\otimes 1}\mathrm{e}^{\mathrm{i}\frac{s_i}{\sqrt{n}}C\otimes 1}\overline{\mathrm{Tr}\prod_{i=1}^n \mathrm{e}^{\frac{1}{n}1\otimes A}\mathrm{e}^{\mathrm{i}\frac{s_i}{\sqrt{n}}1\otimes C}}\mathrm{e}^{-\frac{\sum_{i=1}^n s_i^2}{4}}\cdot\frac{1}{(4\pi)^{n/2}}\int (A\leftrightarrow B).$$

Reversing the above steps yields the claim.

We now write the Heisenberg Hamiltonian as

$$\begin{aligned}H_\Lambda^{(u)}(v) = &\sum_{\{x,y\}\in E_\Lambda}\left((S_x^1-S_y^1)^2+(\sqrt{u}S_x^2-\sqrt{u}S_y^2)^2+((S_x^3+v_x/2)-(S_y^3+v_y/2))^2\right)\\ &+E_\Lambda-\frac{1}{4}\sum_{\{x,y\}\in E_\Lambda}(v_x-v_y)^2.\end{aligned}$$

where $E_\Lambda = -d\sum_{x\in\Lambda}\left((S_x^1)^2+u(S_x^1)^2+(S_x^3)^2\right)$, and the remaing term is removed in the definition of $\tilde{Z}_{\beta,\Lambda}^{(u)}(v)$. The lemma now follows from (6.3) with $\mathcal{H}_\Lambda = \mathcal{H}_{\Lambda_1}\otimes\mathcal{H}_{\Lambda_2}$ and

$$\begin{aligned}A &= -\beta\sum_{\{x,y\}\in E_{\Lambda_1}}\left((S_x^1-S_y^1)^2+(\sqrt{u}S_x^2-\sqrt{u}S_y^2)^2+((S_x^3+v_x/2)-(S_y^3+v_y/2))^2\right)-\beta E_{\Lambda_1}\\ B &= -\beta\sum_{\{x,y\}\in E_{\Lambda_2}}\left((S_x^1-S_y^1)^2+(\sqrt{u}S_x^2-\sqrt{u}S_y^2)^2+((S_x^3+v_x/2)-(S_y^3+v_y/2))^2\right)-\beta E_{\Lambda_2}\\ C_i^1 &= \sqrt{\beta}S_{x_i}^1,\quad D_i^1=\sqrt{\beta}S_{y_i}^1,\quad C_i^2=\sqrt{\beta u}S_{x_i}^2,\quad D_i^1=\sqrt{\beta u}S_{y_i}^2,\\ C_i^3 &= \sqrt{\beta}(S_{x_i}^3+v_{x_i}/2),\quad D_i^3=\sqrt{\beta}(S_{y_i}^3+v_{y_i}/2)\end{aligned}$$

where $\{x_i,y_i\}$ denote the edges crossing the boundary between Λ_1 and Λ_2. Note that S^1, S^3 are real and S^2 is imaginary, so that the above matrices are real for $u\le 0$. $\square$

Proof of Lemma 44. We prove the equivalent statement $\tilde{Z}_{\beta,\Lambda}^{(u)}(v)\le\tilde{Z}_{\beta,\Lambda}^{(u)}(0)$, which can be interpreted as a variational problem, namely $v=0$ is a maximiser of the functional $v\mapsto\tilde{Z}_{\beta,\Lambda}^{(u)}(v)$. Since $\tilde{Z}_{\beta,\Lambda}^{(u)}: l^\infty(\Lambda)\to\mathbb{R}$ is continuous, bounded and $\lim_{\|v\|_\infty\to\infty}\tilde{Z}_{\beta,\Lambda}^{(u)}(v)=0$, there is a maximiser. Let $\bar{v}$ be a maximiser and $\bar{Z}=\tilde{Z}_{\beta,\Lambda}^{(u)}(\bar{v})$. If $\tilde{Z}_{\beta,\Lambda}^{(u)}(\bar{v}_1|R\bar{v}_1)<\bar{Z}$, then Lemma 43 yields $\bar{Z}^2<\bar{Z}\tilde{Z}_{\beta,\Lambda}^{(u)}(R\bar{v}_2|\bar{v}_2)$, namely $\tilde{Z}_{\beta,\Lambda}^{(u)}(R\bar{v}_2|\bar{v}_2)>\bar{Z}$, which is a contradiction. Hence, if $\bar{v}$ is a maximiser, so is $\bar{v}_1|R\bar{v}_1$. Since this holds for any reflection R, this implies inductively that the constant field is a maximiser, and in fact any constant field is so, since $\tilde{Z}_{\beta,\Lambda}^{(u)}(v+\text{const})=\tilde{Z}_{\beta,\Lambda}^{(u)}(v)$. $\square$

Proof of Lemma 45. Lemma 44 implies that $\partial^2/\partial\lambda^2\tilde{Z}_{\beta,\Lambda}^{(u)}(\lambda v)|_{\lambda=0}\le 0$, or equivalently

$$\left(Z_{\beta,\Lambda}^{(u)}(0)\right)^{-1}\frac{\partial^2}{\partial\lambda^2}Z_{\beta,\Lambda}^{(u)}(\lambda v)\Big|_{\lambda=0}\le\frac{\beta}{2}\langle v,-\Delta v\rangle.$$

Since $Z^{(u)}_{\beta,\Lambda}(\lambda v) = \operatorname{Tr}\exp(-\beta(H^{(u)}_{\Lambda} - \lambda\langle S^3, \Delta v\rangle))$, Duhamel's formula $(\exp(F(t))' = \int_0^1 \exp(sF)F' \exp((1-s)F)ds$ yields

$$\frac{\partial^2}{\partial\lambda^2} Z^{(u)}_{\beta,\Lambda}(\lambda v)\Big|_{\lambda=0} = \beta^2 \int_0^1 \operatorname{Tr}\left(\mathrm{e}^{-\beta s H^{(u)}_{\Lambda}} \langle S^3, \Delta v\rangle \mathrm{e}^{-\beta(1-s)H^{(u)}_{\Lambda}} \langle S^3, \Delta v\rangle\right) ds,$$

namely

$$2\beta \left(\langle S^3, -\Delta v\rangle, \langle S^3, -\Delta v\rangle\right)^{(u)}_{\beta} \le \langle v, -\Delta v\rangle,$$

for any field v. Let $v_x(k) = \cos(kx)$, $k \in \Lambda^* \setminus \{0\}$ for which $-\Delta v(k) = E(k)v(k)$. Hence,

$$2\beta E(k) \sum_{x,y\in\Lambda} \cos(kx)\cos(ky)(S^3_x, S^3_y)^{(u)}_{\beta} \le \sum_{x\in\Lambda} \cos^2(kx).$$

It remains to use the translation invariance of $(S^3_x, S^3_y)^{(u)}_{\beta}$ to express the right hand side as

$$\sum_x \cos^2(kx) \sum_z \cos(kz)(S^3_0, S^3_z)^{(u)}_{\beta} + \sum_x \cos(kx)\sin(kz) \sum_z \sin(kz)(S^3_0, S^3_z)^{(u)}_{\beta}.$$

The second term vanishes as $(S^3_0, S^3_z)^{(u)}_{\beta} = (S^3_z, S^3_0)^{(u)}_{\beta}$. Similarly, the sum over z in the first one equals the Fourier transform of $(S^3_0, S^3_\cdot)^{(u)}_{\beta}$, which yields the claim. □

Proof of Lemma 46. This follows from 'Falk-Bruch's inequality', namely

$$\omega^{(u)}_{\beta,\Lambda}(A^*A + AA^*) \le \sqrt{(A^*, A)^{(u)}_{\beta}\,\omega^{(u)}_{\beta,\Lambda}([A^*, [H, A]])} + 2(A^*, A)^{(u)}_{\beta}$$

applied to $A = |\Lambda|^{-1}\sum_x \exp(-\mathrm{i}kx)S^3_x$. Indeed

$$\begin{aligned}
\omega^{(u)}_{\beta,\Lambda}(A^*A + AA^*) &= 2\omega^{(u)}_{\beta,\Lambda}\widehat{(S^3_0, S^3_\cdot)}^{(u)}_{\beta}(k)\\
\omega^{(u)}_{\beta,\Lambda}([A^*[H, A]]) &= 4\beta E^{(u)}(k)\\
(A^*, A)^{(u)}_{\beta} &= \widehat{(S^3_0, S^3_\cdot)}^{(u)}_{\beta}(k)
\end{aligned}$$

and we conclude by the bound on $\widehat{(S^3_0, S^3_\cdot)}^{(u)}_{\beta}(k)$. □