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Problem Sheet 6

Hand-in deadline: 28.06.2017 before 12:15 in the designated MSP box (1st floor, next to the library).

Ex. 1 [Structural stability]: Consider two self-adjoint operators H, V on the Hilbert space $\mathcal{H}$, such that $V \in \mathcal{L}(\mathcal{H})$ and $\exp(-H)$ is trace-class. Let τ_V^t , $t \in \mathbb{R}$, be the automorphism of the observable algebra $\mathcal{A} = \mathcal{L}(\mathcal{H})$ defined by

$$\tau_V^t(A) = \mathrm{e}^{\mathrm{i}t(H+V)} A \mathrm{e}^{-\mathrm{i}t(H+V)}.$$

Let $\omega_{\beta,V}$, $\beta \in (0, \infty)$ be the Gibbs state associated with $H + V$.

The goal of this exercise is to derive a rigorous expansion of $\omega_{\beta,V}$ with respect to V,

$$\omega_{\beta,V}(A) = \omega_{\beta,0}(A) + \sum_{n=1}^{\infty} \nu_n(A), \qquad \beta\|V\| < \ln 2, \tag{1}$$

using the interaction picture propagator

$$U_V(t) = \mathrm{e}^{\mathrm{i}t(H+V)} \mathrm{e}^{-\mathrm{i}tH}.$$

1. Prove that $\exp(-\beta(H + V))$ is a trace-class operator so that $\omega_{\beta,V}$ is well-defined

   Solution: Let $(\psi_k)_{k\in\mathbb{N}}$ be an ONB of $\mathcal{H}$. By the Golden-Thompson inequality (applied to operators on the finite-dimensional Hilbert space $\mathrm{span}\{\psi_1, \ldots, \psi_N\}$), we have

   $$\sum_{k=1}^{N} \langle \psi_k, e^{-\beta(H+V)} \psi_k \rangle \le \sum_{k=1}^{N} \langle \psi_k, e^{-\beta H} e^{-\beta V} \psi_k \rangle$$

   for any $N \in \mathbb{N}$. Since $e^{-\beta H} \in \mathcal{J}_1(\mathcal{H})$ and $e^{-\beta V} \in \mathcal{L}(\mathcal{H})$, it follows that $e^{-\beta H} e^{-\beta V} \in \mathcal{J}_1(\mathcal{H})$. Hence, letting $N \to \infty$, we get

   $$0 \le \mathrm{Tr}(e^{-\beta(H+V)}) \le \mathrm{Tr}(e^{-\beta H} e^{-\beta V}) < \infty.$$

   Since $e^{-\beta(H+V)} \ge 0$, it is in $\mathcal{J}_1(\mathcal{H})$.

2. Check the following basic relations:

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}t(H+V)} &= U_V(t)\mathrm{e}^{\mathrm{i}tH};\\
\tau_V^t(A) &= U_V(t)\tau_0^t(A)U_V(t)^{-1};\\
U_V(t)^{-1} &= U_V(t)^* = \tau_0^t(U_V(-t));\\
U_V(t+s) &= U_V(s)\tau_0^s(U_V(t));\\
\dot{U}_V(t) &= \mathrm{i}U_V(t)\tau_0^t(V), \quad U_V(0) = 1.
\end{aligned}$$

   Solution: Straightforward computation.

3. Use the fact that Dyson's expansion

$$U_V(t) = 1 + \sum_{k=1}^{\infty}(\mathrm{i}t)^k \int_{0\le s_1\le\ldots\le s_k\le 1} \tau^{ts_1}(V)\cdots\tau^{ts_k}(V)\, ds_1\cdots ds_k \tag{2}$$

   is uniformly convergent on compact sets of $\mathbb{C}$ to show that

$$\omega_{\beta,V}(A) = \frac{\omega_{\beta,0}(AU_V(\mathrm{i}\beta))}{\omega_{\beta,0}(U_V(\mathrm{i}\beta))}. \tag{3}$$

   Solution: We assume that V is analytic for τ_t (otherwise it is not clear how $U_V(i\beta)$ is defined). Then the uniform convergence of the series (2) on compact sets implies that $U_V(\cdot)$ has an analytic continuation to $\mathbb{C}$. A straightforward calculation gives

$$\omega_{\beta,V}(A) = \frac{\mathrm{Tr}(e^{-\beta(H+V)}A)}{\mathrm{Tr}(e^{-\beta(H+V)})} = \frac{\mathrm{Tr}(U_V(i\beta)e^{-\beta H}A)}{\mathrm{Tr}(U_V(i\beta)e^{-\beta H})} = \frac{\omega_{\beta,0}(AU_V(\mathrm{i}\beta))}{\omega_{\beta,0}(U_V(\mathrm{i}\beta))}.$$

   We used cyclicity of the trace and the fact that the normalizing factors cancel in the last identity.

4. Use Golden-Thomson's inequality

$$\mathrm{Tr}\left(\mathrm{e}^{A+B}\right) \le \mathrm{Tr}\left(\mathrm{e}^{A}\mathrm{e}^{B}\right)$$

   and Duhamel's formula to show that for any $\alpha \in \mathbb{C}$,

$$|\omega_{\beta,0}(U_{\alpha V}(\mathrm{i}\beta)) - 1| \le \mathrm{e}^{|\alpha\beta|\|V\|} - 1. \tag{4}$$

   Solution: Without loss of generality we may assume $\mathrm{Tr}(e^{-\beta H}) = 1$. Then

$$\omega_{\beta,0}(U_{\alpha V}(\mathrm{i}\beta)) - 1 = \mathrm{Tr}((U_V(\mathrm{i}\beta) - \mathbf{1})e^{-\beta H}).$$

Using Duhamel's formula, we have

$$(U_V(\mathrm{i}\beta) - \mathbf{1})e^{-\beta H} = -\int_0^\beta e^{-s(H+V)} V e^{(s-\beta)H} ds,$$

so that

$$\begin{aligned}
|\operatorname{Tr}((U_V(\mathrm{i}\beta) - \mathbf{1})e^{-\beta H})| &\leq \int_0^\beta |\operatorname{Tr}(e^{-s(H+V)} V e^{(s-\beta)H})| ds \\
&\leq \|V\| \int_0^\beta \|e^{(s-\beta)H}\| \operatorname{Tr}(e^{-s(H+V)}) ds \\
&\leq \|V\| \int_0^\beta \operatorname{Tr}(e^{-sH} e^{-sV}) ds \\
&\leq \|V\| \int_0^\beta \operatorname{Tr}(e^{-sH}) e^{s\|V\|} ds \\
&\leq \|V\| \int_0^\beta e^{s\|V\|} ds \\
&= e^{\beta\|V\|} - 1
\end{aligned}$$

The same of course holds for αV.

5. Use (3) and (4) to prove that $\alpha \mapsto \omega_{\beta,\alpha V}(A)$ is analytic at 0 with

$$\omega_{\beta,\alpha V}(A) = \sum_{n=0}^\infty \alpha^n \nu_n(A), \qquad |\alpha| < \frac{\log 2}{\beta\|V\|}. \tag{5}$$

Solution: Analyticity of $\alpha \mapsto U_{\alpha V}(i\beta)$ follows from (2). Since by (4),

$$|\omega_{\beta,\alpha V}(U_{\alpha V}(i\beta))| \geq 1 - |\omega_{\beta,\alpha V}(U_{\alpha V}(i\beta)) - 1| \geq 1 - (e^{|\alpha|\beta\|V\|} - 1),$$

the claim follows.

6. Use (2,3,5) to conclude that

$$\begin{aligned}
&\nu_0(A) = \omega_{\beta,0}(A), \\
&\nu_1(A) = -\beta \int_0^1 \left[\omega_{\beta,0}\left(A\tau^{\mathrm{i}\beta s}(V)\right) - \omega_{\beta,0}(A)\omega_{\beta,0}\left(\tau^{\mathrm{i}\beta s}(V)\right)\right] \mathrm{d}s.
\end{aligned}$$

Solution: This follows from analyticity and a first order Taylor expansion.

Ex. 2 [Absence of symmetry breaking in 1D]: Consider a one-dimensional quantum spin chain. For each $x \in \mathbb{Z}$, let $\mathcal{H}_x \simeq \mathbb{C}^n$ for a fixed $n \geqslant 2$, and let $\mathcal{A}_\mathbb{Z}$ be the usual quasi-local algebra built upon $\mathcal{A}_{\{x\}} = \mathcal{L}(\mathcal{H}_x)$, namely:

$$\mathcal{A}_\Lambda = \otimes_{x\in\Lambda}\mathcal{A}_{\{x\}}, \qquad \mathcal{A}_{\mathrm{loc}} = \bigcup_{\Lambda\subset\mathbb{Z}:|\Lambda|<\infty} \mathcal{A}_\Lambda, \qquad \mathcal{A}_\mathbb{Z} = \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}.$$

Let $(\Lambda_m)_{m\in\mathbb{N}}$ be the sequence $\Lambda_m = [-m, m] \cap \mathbb{Z}$. Consider:

- unitary elements $U_x \in \mathcal{A}_{\{x\}}$ and the associated map

$$\alpha_\Lambda(A) := (\otimes_{x\in\Lambda}U_x^*)\, A\, (\otimes_{y\in\Lambda}U_y)\,, \qquad A \in \mathcal{A}_\Lambda;$$

- the local Hamiltonian $H_\Lambda \in \mathcal{A}_\Lambda$, given by a two-body interaction

$$H_\Lambda = \sum_{x,y\in\Lambda} J(x,y)\Phi_{x,y}\,,$$

  where $\Phi_{x,y} \in \mathcal{A}_{\{x\}\cup\{y\}}$ and $\|\Phi_{x,y}\| \leqslant 1$; we shall assume that the associated dynamics τ^t exists on $\mathcal{A}_\mathbb{Z}$.

1. Prove that there exists an automorphism α of $\mathcal{A}_\mathbb{Z}$ such that

$$\lim_{m\to\infty} \alpha_{\Lambda_m}(A) = \alpha(A), \qquad A \in \mathcal{A}_\mathbb{Z}.$$

   Solution: Since $\mathcal{A}_{\mathrm{loc}}$ is dense in $\mathcal{A}_\mathbb{Z}$ (by definition) it is sufficient to prove the existence of the limit for $A \in \mathcal{A}_{\mathrm{loc}}$. Hence let $A \in \mathcal{A}_{\mathrm{loc}}$ and let $\epsilon > 0$ be arbitrary. There exists Λ^* such that $A \in \mathcal{A}_{\Lambda^*}$. Thus, if m_0 is so large that $[-m_0, m_0] \supset \Lambda^*$, it follows that

$$\alpha_{\Lambda_m}(A) = (\otimes_{x\in\Lambda^*}U_x^*)\, A\, (\otimes_{y\in\Lambda^*}U_y)\,, \quad m \geq m_0.$$

   Hence the sequence becomes constant for large m and the limit trivially exists.

2. Prove that if $\alpha(\Phi_{x,y}) = \Phi_{x,y}$ for all $(x,y) \in \mathbb{Z}\times\mathbb{Z}$, then α is a symmetry of the dynamics.

   Solution:

3. Finally, assume that there exists $C < \infty$ such that

$$\sup_{x\in\mathbb{Z}} \sum_{y\in\mathbb{Z}} |J(x,y)||x-y| = C.$$

   Prove that if ω is a (τ, β)-KMS state for a $\beta \in (0,\infty)$, then $\omega \circ \alpha = \omega$.

   Solution: 1. First, we note that for any $\Lambda \in \mathcal{F}(\mathbb{Z})$, the map α_Λ is a *-automorphism of $\mathcal{A}_\Lambda$. Indeed, for any $A, B \in \mathcal{A}_\Lambda$,

(a) Clearly, $\alpha_\Lambda(A+\lambda B) = \alpha_\Lambda(A) + \lambda\alpha(B)$

(b) Since $\otimes_{x\in\Lambda}U_x$ is unitary, $\alpha_\Lambda(AB) = (\otimes_{x\in\Lambda}U_x)^* A(\otimes_{x\in\Lambda}U_x)(\otimes_{x\in\Lambda}U_x)^* B(\otimes_{x\in\Lambda}U_x) = \alpha_\Lambda(A)\alpha_\Lambda(B)$

(c) Again by unitarity, we have that $\|\alpha_\Lambda(A)\| = \|A\|$

(d) Finally, $\alpha_\Lambda(A^*) = \alpha_\Lambda(A)^*$

In particular, α_Λ is a bounded linear map on the dense subalgebra $\mathcal{A}_{\mathrm{loc}}$ with bound 1. Hence, it extends to a bounded linear map with the same bound on $\mathcal{A}_\mathbb{Z} = \overline{\mathcal{A}_{\mathrm{loc}}}$. In particular, if $A \in \mathcal{A}_\Lambda$, there is m_0 such that $\forall m \geq m_0$, $\Lambda_m \supset \Lambda$ so that $\alpha_{\Lambda_m}(A) = \alpha_\Lambda(A) = \alpha(A)$ whenever $m \geq m_0$ and hence the convergence.

Points 2 and 4 above follow by density. We consider 4, point 2 follows similarly. For any $A \in \mathcal{A}_\mathbb{Z}$, there is a sequence $A_n \in \mathcal{A}_{\mathrm{loc}}$ and $\Lambda^{(n)} \in \mathcal{F}(\mathbb{Z})$ such that $A_n \in \Lambda^{(n)}$ and $A_n \to A$ in the C* norm. We then have

$$\alpha(A^*) = \lim_{n\to\infty}\alpha(A_n^*) = \lim_{n\to\infty}\alpha_{\Lambda^{(n)}}(A_n^*) = \lim_{n\to\infty}\alpha_{\Lambda^{(n)}}(A_n)^* = \lim_{n\to\infty}\alpha(A_n)^* = \alpha(A)^*$$

where we used the continuity of $A \mapsto \alpha(A)$ in the first and last equalities.

2. We only need to check the conditions of the relevant theorem stated in class. By the above, the unitaries $U_m = \otimes_{x\in\Lambda_m}U_x \in \mathcal{A}_\mathbb{Z}$ are such that

$$\lim_{m\to\infty}\|\alpha(A) - U_m^* A U_m\| = 0, \qquad \text{for all } A \in \mathcal{A}_\mathbb{Z}.$$

Now, for $A \in \mathcal{A}_\Lambda$, we have that $\delta(A) = \mathrm{i}\sum_{\substack{(x,y)\in\mathbb{Z}\times\mathbb{Z}:\\ \{x,y\}\cap\Lambda\neq\emptyset}} J(x,y)[\Phi_{x,y}, A]$, and since $\|\Phi_{x,y}\| \leq 1$,

$$\|\delta(A)\| \leq 2\|A\||\Lambda|\sup_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}}|J(x,y)| < \infty$$

so that $\mathcal{A}_{\mathrm{loc}} \subset D(\delta)$. In particular, $U_m \in \mathcal{A}_{\Lambda_m} \subset D(\delta)$.

Since $\alpha(\Phi_{x,y}) = \Phi_{x,y}$, we have that $[\Phi_{x,y}, U_m] = (\Phi_{x,y} - U_m\Phi_{x,y}U_m^*)U_m = 0$ if $x, y \in \Lambda_m$ so that

$$\|\delta(U_m)\| \leq \sum_{\substack{(x,y)\in\mathbb{Z}\times\mathbb{Z}:\\ x\in\Lambda_m, y\in\Lambda_m^c}} |J(x,y)|\|[\Phi_{x,y}, U_m]\| \leq 2\sum_{\substack{(x,y)\in\mathbb{Z}\times\mathbb{Z}:\\ x\in\Lambda_m, y\in\Lambda_m^c}} |J(x,y)|$$

where we used $\|U_m\| = 1$. Reorganizing the sum, we obtain

$$\|\delta(U_m)\| \leq 2\sum_{d\geq 2}\sum_{\substack{(x,y)\in\mathbb{Z}\times\mathbb{Z}:\\ x\in\Lambda_m, y\in\Lambda_m^c, |x-y|=d}} |J(x,y)|$$

and it remains to note that there are exactly $2 \cdot \min\{(d-1), m\} \leq 2d$ terms in the second sum to get the bound

$$\|\delta(U_m)\| \leq 4 \sum_{d\geq 2}(d-1) \sup_{(x,y):|x-y|=d} |J(x,y)| \leq 4 \cdot \sup_{x\in\mathbb{Z}} \sum_{y\in\mathbb{Z}} |J(x,y)||x-y| = 4C$$

Since this bound is uniform in m, the theorem ensures that for any (τ^t, β)-KMS state, $\omega \circ \alpha = \omega$.