Problem Sheet 6

Hand-in deadline: 28.06.2017 before 12:15 in the designated MSP box (1st floor, next to the library).

Ex. 1 [Structural stability]: Consider two self-adjoint operators H, V on the Hilbert space $\mathcal{H}$, such that $V \in \mathcal{L}(\mathcal{H})$ and $\exp(-H)$ is trace-class. Let τ_V^t , $t \in \mathbb{R}$, be the automorphism of the observable algebra $\mathcal{A} = \mathcal{L}(\mathcal{H})$ defined by

$$\tau_V^t(A) = \mathrm{e}^{\mathrm{i}t(H+V)} A \mathrm{e}^{-\mathrm{i}t(H+V)}.$$

Let $\omega_{\beta,V}$, $\beta \in (0,\infty)$ be the Gibbs state associated with $H + V$.

The goal of this exercise is to derive a rigorous expansion of $\omega_{\beta,V}$ with respect to V,

$$\omega_{\beta,V}(A) = \omega_{\beta,0}(A) + \sum_{n=1}^{\infty} \nu_n(A), \qquad \beta\|V\| < \ln 2, \tag{1}$$

using the interaction picture propagator

$$U_V(t) = \mathrm{e}^{\mathrm{i}t(H+V)} \mathrm{e}^{-\mathrm{i}tH}.$$

1. Prove that $\exp(-\beta(H + V))$ is a trace-class operator so that $\omega_{\beta,V}$ is well-defined
2. Check the following basic relations:
$$\begin{aligned} \mathrm{e}^{\mathrm{i}t(H+V)} &= U_V(t)\mathrm{e}^{\mathrm{i}tH}; \\ \tau_V^t(A) &= U_V(t)\tau_0^t(A)U_V(t)^{-1}; \\ U_V(t)^{-1} &= U_V(t)^* = \tau_0^t(U_V(-t)); \\ U_V(t+s) &= U_V(s)\tau_0^s(U_V(t)); \\ \dot{U}_V(t) &= \mathrm{i}U_V(t)\tau_0^t(V), \quad U_V(0) = 1. \end{aligned}$$
3. Use the fact that Dyson's expansion
$$U_V(t) = 1 + \sum_{k=1}^{\infty} (\mathrm{i}t)^k \int_{0\le s_1 \le \ldots \le s_k \le 1} \tau^{ts_1}(V) \cdots \tau^{ts_k}(V)\, ds_1 \cdots ds_k \tag{2}$$
is uniformly convergent on compact sets of $\mathbb{C}$ to show that
$$\omega_{\beta,V}(A) = \frac{\omega_{\beta,0}(AU_V(\mathrm{i}\beta))}{\omega_{\beta,0}(U_V(\mathrm{i}\beta))}. \tag{3}$$

4. Use Golden-Thomson's inequality
$$\mathrm{Tr}\left(\mathrm{e}^{A+B}\right) \leq \mathrm{Tr}\left(\mathrm{e}^{A}\mathrm{e}^{B}\right)$$
and Duhamel's formula to show that for any $\alpha \in \mathbb{C}$,
$$|\omega_{\beta,0}(U_{\alpha V}(\mathrm{i}\beta)) - 1| \leq \mathrm{e}^{|\alpha\beta|\|V\|} - 1. \tag{4}$$

5. Use (3) and (4) to prove that $\alpha \mapsto \omega_{\beta,\alpha V}(A)$ is analytic at 0 with
$$\omega_{\beta,\alpha V}(A) = \sum_{n=0}^{\infty} \alpha^n \nu_n(A), \qquad |\alpha| < \frac{\log 2}{\beta\|V\|}. \tag{5}$$

6. Use (2,3,5) to conclude that
$$\begin{aligned} \nu_0(A) &= \omega_{\beta,0}(A), \\ \nu_1(A) &= -\beta \int_0^1 \left[\omega_{\beta,0}\left(A\tau^{\mathrm{i}\beta s}(V)\right) - \omega_{\beta,0}(A)\omega_{\beta,0}\left(\tau^{\mathrm{i}\beta s}(V)\right)\right] \mathrm{d}s. \end{aligned}$$

Ex. 2 [Absence of symmetry breaking in 1D]: Consider a one-dimensional quantum spin chain. For each $x \in \mathbb{Z}$, let $\mathcal{H}_x \simeq \mathbb{C}^n$ for a fixed $n \geqslant 2$, and let $\mathcal{A}_{\mathbb{Z}}$ be the usual quasi-local algebra built upon $\mathcal{A}_{\{x\}} = \mathcal{L}(\mathcal{H}_x)$, namely:
$$\mathcal{A}_\Lambda = \otimes_{x\in\Lambda}\mathcal{A}_{\{x\}}, \qquad \mathcal{A}_{\mathrm{loc}} = \bigcup_{\Lambda\subset\mathbb{Z}:|\Lambda|<\infty} \mathcal{A}_\Lambda, \qquad \mathcal{A}_{\mathbb{Z}} = \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}.$$
Let $(\Lambda_m)_{m\in\mathbb{N}}$ be the sequence $\Lambda_m = [-m, m] \cap \mathbb{Z}$. Consider:

- unitary elements $U_x \in \mathcal{A}_{\{x\}}$ and the associated map
$$\alpha_\Lambda(A) := \left(\otimes_{x\in\Lambda}U_x^*\right) A \left(\otimes_{y\in\Lambda}U_y\right), \qquad A \in \mathcal{A}_\Lambda;$$
- the local Hamiltonian $H_\Lambda \in \mathcal{A}_\Lambda$, given by a two-body interaction
$$H_\Lambda = \sum_{x,y\in\Lambda} J(x,y)\Phi_{x,y}\,,$$
where $\Phi_{x,y} \in \mathcal{A}_{\{x\}\cup\{y\}}$ and $\|\Phi_{x,y}\| \leqslant 1$; we shall assume that the associated dynamics τ^t exists on $\mathcal{A}_{\mathbb{Z}}$.

1. Prove that there exists an automorphism α of $\mathcal{A}_{\mathbb{Z}}$ such that
$$\lim_{m\to\infty} \alpha_{\Lambda_m}(A) = \alpha(A), \qquad A \in \mathcal{A}_{\mathbb{Z}}.$$

2. Prove that if $\alpha(\Phi_{x,y}) = \Phi_{x,y}$ for all $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, then α is a symmetry of the dynamics

3. Finally, assume that there exists $C < \infty$ such that

$$\sup_{x\in\mathbb{Z}} \sum_{y\in\mathbb{Z}} |J(x, y)||x - y| = C.$$

   Prove that if ω is a (τ, β)-KMS state for a $\beta \in (0, \infty)$, then $\omega \circ \alpha = \omega$.