Problem Sheet 5

Hand-in deadline: 21.06.2017 before 12:15 in the designated MSP box (1st floor, next to the library).

Ex. 1: Let $(\tau_t)_{t\in\mathbb{R}}$ be a weakly continuous group of $*$-automorphisms of a C^*-algebra $\mathcal{A}$ with unit. Recall that this means that

- τ_t is a $*$-automorphisms of $\mathcal{A}$ for each $t \in \mathbb{R}$ and $\tau_0 = \mathbf{1}$ (the identity)
- For each $t, s \in \mathbb{R}$: , $\tau_t\tau_s = \tau_{t+s}$, $\tau_{-t} = \tau_t^{-1}$,
- For every state ρ on $\mathcal{A}$ and every $A \in \mathcal{A}$, $\lim_{t\to 0} \rho(\tau_t(A)) = \rho(A)$.

Assume now that there is a state $\omega \in \mathcal{A}$ that is invariant with respect to τ_t, namely such that

$$\omega(\tau_t(A)) = \omega(A), \quad A \in \mathcal{A}, t \in \mathbb{R}.$$

Denote by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the corresponding GNS representation. Prove that there exists a densely defined self-adjoint operator H on the Hilbert space $\mathcal{H}_\omega$ such that

$$\pi_\omega(\tau_t(A)) = e^{itH}\pi_\omega(A)e^{-itH}, \quad A \in \mathcal{A}, t \in \mathbb{R},$$

Ω_ω is in the domain of H and $H\Omega_\omega = 0$.

Ex. 2: Let $(\tau_t^n)_{t\in\mathbb{R}}$, $n \in \mathbb{N}$, be a sequence of dynamics on a C^*-algebra $\mathcal{A}$ with generators δ_n such that $\mathcal{D}(\delta_n) \subset \mathcal{D}$ for all $n \in \mathbb{N}$, where $\mathcal{D} \subset \mathcal{A}$ is dense. Assume that $\tau_t^n(A)$ has a limit $\tau_t(A)$ as $n \to \infty$, for each $A \in \mathcal{A}$ and $t \in \mathbb{R}$, and that τ_t is again a dynamics with generator δ and $\mathcal{D}(\delta) \subset \mathcal{D}$.

Let $(\omega^n)_{n\in\mathbb{N}}$ be a sequence of (τ^n, β)-KMS states. Prove the existence of a (τ, β)-KMS state.

Ex. 3: Let $(\mathcal{A}, \tau_t)$ be a C^*-dynamical system and let ω be a state on $\mathcal{A}$. For $\beta \in \mathbb{R}$ prove the equivalence of the following two statements:

(A) ω is a (τ, β)-KMS state.

(B) The relation

$$\int_{\mathbb{R}} f(t)\omega(A\tau_t(B))dt = \int_{\mathbb{R}} f(t+i\beta)\omega(\tau_t(B)A)dt$$

is valid for all $A, B \in \mathcal{A}$ and f with $\check{f} \in C_c^\infty(\mathbb{R})$.

Hint: Use the Paley-Wiener theorem which states that an entire analytic function g is the Fourier transform of a C_c^∞ function with support in $[-R, R]$ if and only if for each N there exists C_N such that

$$|g(\xi)| \leq \frac{C_N e^{R|\mathrm{Im}\xi|}}{(1+|\xi|)^N}, \quad \xi \in \mathbb{C}.$$

Ex. 4: Let $(\mathcal{A}, \tau_0^t)$ be a C^*-dynamical system with generator δ_0. Let V be a self-adjoint element of $\mathcal{A}$. Let $\delta_V = \delta_0 + \mathrm{i}[V, \cdot]$, and let τ_V^t be the corresponding dynamics.

i) Prove that for any $t \in \mathbb{R}$ and $A \in \mathcal{A}$, the *Dyson-Robinson expansion*

$$\begin{aligned}\tau_V^t(A) &= \tau_0^t(A) \\ &\quad + \sum_{N=1}^{\infty} \mathrm{i}^N \int_{0 \leq t_1 \leq \ldots \leq t_N \leq t} [\tau_0^{t_1}(V), [\ldots, [\tau_0^{t_n}(V), \tau_0^t(A)] \ldots]] dt_1 \ldots dt_n\end{aligned}$$

holds, the series being norm-convergent.

ii) Let $\Gamma_V(t)$ be defined by the Ansatz $\tau_V^t(A)\Gamma_V(t) = \Gamma_V(t)\tau_0^t(A)$ for $A \in \mathcal{A}, t \in \mathbb{R}$. Prove that $\Gamma_V(t)$ solves the differential equation

$$-\mathrm{i}\dot{\Gamma}_V(t) = \Gamma_V(t)\tau_0^t(V), \quad \Gamma_V(0) = \mathbf{1}.$$

iii) Prove that $\Gamma_V(t)$ satisfies the *cocycle relations*

$$\Gamma_V(t+s) = \Gamma_V(t)\tau_0^t(\Gamma_V(s)) = \tau_V^t(\Gamma_V(s))\Gamma_V(t).$$