Problem Sheet 4

Hand-in deadline: 24.05.2017 before 12:15 in the designated MSP box (1st floor, next to the library).

Ex. 1: Let $(\mathcal{A}, \tau_t)$ be a C^*-dynamical system and let δ be its generator. For any $A \in \mathcal{A}$, $m \in \mathbb{N}$, let

$$A_m := \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}} \tau_t(A) \exp(-mt^2) dt.$$

Prove that

(a) A_m is analytic for τ_t, i.e. the map $t \mapsto \tau_t(A)$ has an analytic continuation to $\mathbb{C}$;

(c) The $*$-subalgebra $\{A_m : A \in \mathcal{A}, m \in \mathbb{N}\}$ is dense in $\mathcal{A}$.

Ex. 2: Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $-\Delta_\Omega^D$ be the Dirichlet Laplacian on Ω. Then *Weyl's law* says that

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{\omega_d}{(4\pi)^{d/2}} |\Omega|. \tag{1}$$

Here $N(\lambda) = \#\{j \in \mathbb{N} : E_j \leq \lambda\}$ is the counting function, $0 < E_0 \leq E_1 \leq E_2 \leq \dots$ are the eigenvalues of $-\Delta_\Omega^D$ and ω_d is the volume of the unit ball in $\mathbb{R}^d$. The purpose of this exercise is to prove the upper bound in (1). For simplicity we assume that $\partial\Omega$ is smooth. This implies, in particular, that the eigenfunctions e_j of $-\Delta_\Omega^D$ are in $C^\infty(\overline{\Omega})$ and $e_j = 0$ on $\partial\Omega$.

(a) For $\psi \in L^2(\Omega)$ and $t > 0$ let

$$\mathrm{e}^{t\Delta_\Omega^D} \psi = \sum_{j=1}^{\infty} \mathrm{e}^{-tE_j} \langle e_j, \psi \rangle e_j.$$

be the heat semigroup generated by the Dirichlet Laplacian and denote its kernel by $\widetilde{k}_t(x, y)$. You can use the fact that

$$0 \leq \widetilde{k}_t(x, y) \leq k_t^0(x, y), \quad x, y \in \Omega,$$

where $k_t^0(x,y)$ is the Gaussian kernel

$$k_t^0(x,y) = (4\pi t)^{-d/2} \mathrm{e}^{-|x-y|^2/(4t)}.$$

Prove that

$$|e_j(x)| \leq c E_j^{d/4}$$

for some constant c independent of j.

(b) Prove that

$$\int_\Omega k_t(x,x) = \sum_{j=1}^{\infty} \mathrm{e}^{-E_j t}$$

Hint: You may freely use *Mercer's theorem:* If a nonnegative, bounded selfadjoint operator A on $L^2(\omega)$ has continuous integral kernel $a(\cdot,\cdot)$, then $\mathrm{Tr}(A) = \int_\Omega a(x,x)dx$.

(c) Prove that Weyl's law (1) is equivalent to

$$\lim_{t\to 0} t^{d/2} \int_\Omega k_t(x,x)dx = \frac{|\Omega|}{(4\pi)^{d/2}}.$$

Hint: You may freely use *Karamata's Tauberian theorem*: Let $(E_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers such that the series $\sum_{j\in\mathbb{N}} \mathrm{e}^{-E_j t}$ converges for every $t > 0$. Then for $r > 0$ and $a \in \mathbb{R}$ the following are equivalent.

(i) $\lim_{t\to 0} t^r \sum_{j\in\mathbb{N}} \mathrm{e}^{-E_j t} = a$

(ii) $\lim_{\lambda\to\infty} \lambda^{-r} N(\lambda) = \frac{a}{\Gamma(r+1)}$

(d) Prove the upper bound in Weyl's law (1).

Ex. 3: Consider a bounded domain $\Omega \subset \mathbb{R}^d$ and let $\Omega_L = L\Omega$ for any $L > 0$. Let $-\Delta_L$ be the Dirichlet Laplacian on Ω_L[1]. For $\beta > 0$ and $\mu < \inf \sigma(-\Delta_L)$, define

$$\rho_L(\mu,\beta) := \frac{1}{|\Omega_L|} \mathrm{Tr} \frac{\exp(-\beta(-\Delta_L - \mu))}{1 - z\exp(-\beta(-\Delta_L - \mu))} = \sum_{j=1}^{\infty} \rho_L^{(j)}(\mu,\beta),$$

where

$$\rho_L^{(n)}(\mu,\beta) := \frac{1}{|\Omega_L|} \frac{\exp(-\beta(L^{-2}E_j \mu))}{1 - z\exp(-\beta(L^{-2}E_j - \mu))}.$$

[1] You may use that the lowest eigenvalue of $-\Delta_L$ is nondegenerate and strictly positive.

(a) Let $\overline{\rho} > 0$ and $\beta > 0$ be fixed. Prove that the equation $\overline{\rho} = \rho_L(\mu_L, \beta)$ has a unique solution $\mu_L \in (-\infty, \inf \sigma(-\Delta_L))$.

(b) Prove that

$$\lim_{L\to\infty} \rho_L^{(j)}(\mu_L, \beta) = 0, \quad j > 1.$$

(c) Prove that

$$\sum_{j=2}^{\infty} \rho_L^{(j)}(\mu_L, \beta) \leq C < \infty$$

where the constant C is *independent* of $\overline{\rho}$ (recall that in class it was claimed that $C = \rho_c(\beta)$), and thus

$$\lim_{L\to\infty} \rho_L^{(j)}(\mu_L, \beta) > 0, \quad j = 1$$

if $\overline{\rho} > C$.