Mathematisches Institut der Universität München

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Problem Sheet 3

Hand-in deadline: 17.05.2017 before 12:15 in the designated MSP box (1st floor, next to the library).

Ex. 1: The Fermi sea. Let $0 < \beta < \infty, \mu \in \mathbb{R}$, and let $\omega_{\beta,\mu}$ be the thermal equilibrium state over $\mathrm{CAR}(L^2(\mathbb{R}^d))$. Prove that

$$\omega_{\beta,\mu} \rightharpoonup^* \omega_{\infty,\mu}$$

as $\beta \to \infty$, where

$$\omega_{\infty,\mu}(a^*(\phi_1)\cdots a^*(\phi_n)a(\psi_m)\cdots a(\psi_1)) = \delta_{n,m}\det D(\underline{\Phi},\underline{\Psi}),$$

and

$$D(\underline{\Phi},\underline{\Psi})_{i,j} = \frac{1}{(2\pi)^{d/2}}\int_{|\xi|\le\mu} \overline{\widehat{\psi}_i}(\xi)\widehat{\phi}_j(\xi)d\xi.$$

Note: Argue carefully that the convergence on the monomials above implies by density the convergence on any element of the algebra.

Ex. 2: Araki-Wyss representation. Let $\rho \in \mathcal{B}(\mathcal{H})$ be such that $0 \le \rho = \rho^* \le 1$, and let ω_ρ be the state on $\mathrm{CAR}(\mathcal{H})$ be defined by

$$\omega_\rho(a^*(\phi_1)\cdots a^*(\phi_n)a(\psi_m)\cdots a(\psi_1)) = \delta_{n,m}\det\left((\langle\psi_i,\rho\phi_j\rangle)_{i,j=1}^n\right).$$

Prove that the triple $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ given by $\mathcal{H}_\rho = \mathcal{F}_{\mathrm{a}}(\mathcal{H})\otimes\mathcal{F}_{\mathrm{a}}(\mathcal{H})$, $\Omega_\rho = \Omega\otimes\Omega$ and

$$\pi_\rho(a(\phi)) = a(\sqrt{1-\rho}\,\phi)\otimes 1 + \exp(\mathrm{i}\pi\mathcal{N})\otimes a^*(\sqrt{\rho}\phi)$$

is the GNS triple for ω_ρ, namely:

(i) π_ρ is a representation of $\mathrm{CAR}(\mathcal{H})$

(ii) $\omega_\rho(A) = \langle\Omega_\rho, \pi_\rho(A)\Omega_\rho\rangle$ for all $A \in \mathrm{CAR}(\mathcal{H})$.

Ex. 3: Rate of convergence to TDL. Let

$$\rho^L(\beta, z) := L^{-d} \sum_{n\in\mathbb{N}} \omega^L_{\beta,\mu}(a^*(\phi_n)a(\phi_n))$$

be the density of the Gibbs state on $\mathrm{CAR}(\mathcal{H}_L)$ associated with the Dirichlet Laplacian on $\mathcal{H}_L = L^2([-L/2, L/2]^d)$, where $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_L$. Let

$$\rho(\beta, z) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{z\exp(-\beta|\xi|^2)}{1 + z\exp(-\beta|\xi|^2)} d\xi$$

be the corresponding density in the thermodynamic limit. Prove that

$$0 \leq \frac{\rho(\beta, z) - \rho^L(\beta, z)}{\rho(\beta, z)} \leq C_z \frac{\beta^{1/2}}{L}.$$

Note: For helium at room temperature, this relative error is of order 10^{-7} for L as small as 1cm.

Ex. 4: Schrödinger representation. Let $\pi_S : \mathrm{CCR}(\mathbb{C}) \to \mathcal{L}(L^2(\mathbb{R}))$ be defined by

$$W(z) \longmapsto \pi_S(W(z)) = \mathrm{e}^{\frac{\mathrm{i}}{2}st}U(s)V(t), \qquad z = s + \mathrm{i}t, \quad s, t \in \mathbb{R},$$

where

$$(U(s)\psi)(x) := \mathrm{e}^{\mathrm{i}sx}\psi(x), \qquad (V(t)\psi)(x) := \psi(x + t)$$

are strongly continuous one-parameter unitary groups on $L^2(\mathbb{R})$.

(i) Prove that π_S is a representation of $\mathrm{CCR}(\mathbb{C})$

(ii) Prove that the representation is regular, namely $\mathbb{R} \ni \alpha \mapsto \pi_S(W(\alpha z))$ is strongly continuous for all $z \in \mathbb{C}$

(iii) Let Φ, resp. Π, be the generator of $\mathbb{R} \ni s \mapsto \pi_S(W(s))$, resp. $\mathbb{R} \ni t \mapsto \pi_S(W(\mathrm{i}t))$. If

$$a_S := 2^{-1/2}(\Phi + \mathrm{i}\Pi), \quad a_S^* := 2^{-1/2}(\Phi - \mathrm{i}\Pi))$$

prove that

$$a_S a_S^* \psi - a_S^* a_S \psi = \psi$$

for any $\psi \in \mathcal{S}(\mathbb{R})$.
Hint: What are Φ, Π?

(iv) Prove that the representation is topologically irreducible, namely $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ that are invariant under $\{\pi_S(W(z)) : z \in \mathbb{C}\}$
Hint: Argue by contradiction with a proper invariant subspace $\mathcal{H}_1$ and show that if $\phi \perp \mathcal{H}_1$, then $\phi = 0$