Problem Sheet 2

Hand-in deadline: 10.05.2017 before 12:15 in the designated MSP box (1st floor, next to the library).

Ex. 1: Let $\mathcal{A}$ be a C*-algebra with a unit, denoted 1, and ω be a state, i.e. a positive linear functional over $\mathcal{A}$ such that $\omega(1) = 1$.

(i) Prove the following identities:

$$\omega(A^*) = \overline{\omega(A)}, \qquad |\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B),$$
$$|\omega(A^*BA)| \leq \omega(A^*A)\|B\|$$

(Hint: consider the quadratic form $\lambda \mapsto \omega((A+\lambda B)^*(A+\lambda B)))$

(ii) Let $\mathcal{N} := \{A \in \mathcal{A} : \omega(A^*A) = 0\}$. Prove that $A \in \mathcal{A}, N \in \mathcal{N}$ implies $AN \in \mathcal{N}$, i.e. $\mathcal{N}$ is a left ideal

(iii) Let $h := \mathcal{A}/\mathcal{N}$, and denote ψ_A the equivalence class of $A \in \mathcal{A}$, namely $\psi_A := \{\tilde{A} \in \mathcal{A} : \exists N \in \mathcal{N} : \tilde{A} = A + N\}$. Prove that the bilinear form over h

$$(\psi_A, \psi_B) \longmapsto \langle \psi_A, \psi_B \rangle := \omega(A^*B)$$

is well-defined (i.e. the r.h.s. is independent of the chosen representative of the classes ψ_A, ψ_B) and defines a scalar product

(iv) Hence h equipped with $\langle \cdot, \cdot \rangle$ is a pre-Hilbert space. Let $\mathcal{H}$ denote its completion. Prove that the linear map $\pi : \mathcal{A} \to \mathcal{L}(h)$ defined by

$$\pi(A)\psi_B := \psi_{AB}$$

is bounded, and that it is a *-homomorphism, namely

$$\pi(A^*) = \pi(A)^*, \qquad \pi(AB) = \pi(A)\pi(B).$$

(v) Finally, let $\mathcal{H} \ni \Omega := \psi_1$, where $1 \in \mathcal{A}$. Prove that

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle.$$

The triple $(\mathcal{H}, \pi, \Omega)$ is the *GNS representation* of $\mathcal{A}$ associated with ω.

Ex. 2: Same assumptions as in Ex. 1

(i) Prove that the GNS representation is unique up to unitary equivalence. More precisely, prove that, given any two representations $(\mathcal{H}_i, \pi : i, \Omega_i)$, $i = 1, 2$, the map $U : \mathcal{H}_1 \to \mathcal{H}_2$, given by $U\pi_1(A)\Omega_1 := \pi_2(A)\Omega_2$ for any $A \in \mathcal{A}$ (and extended by linearity and density to all of $\mathcal{H}_1$) is unitary.

(ii) Let α be a $*$-automorphism of $\mathcal{A}$ and suppose that $\omega \circ \alpha = \omega$. Prove that there is a unique unitary operator U on the GNS Hilbert space $\mathcal{H}_\omega$ such that $U\pi_\omega(A) = \pi_\omega(\alpha(A))U$ for $A \in \mathcal{A}$ and $U\Omega_\omega = \Omega_\omega$.

Ex. 3: Consider the C^*-algebra $\mathcal{A} = \mathcal{L}(\mathcal{H})$ of bounded linear operators on a given Hilbert space $\mathcal{H}$. Let $\rho : \mathcal{H} \to \mathcal{H}$ be a strictly positive density matrix, i.e. $\rho = \rho^*$, $0 < \rho \leq \mathbf{1}$, $\mathrm{Tr}_{\mathcal{H}}\, \rho = 1$. Let $\omega : \mathcal{A} \to \mathbb{C}$ be the state realised by ρ, i.e. $\omega(A) := \mathrm{Tr}(\rho A)$ for $A \in \mathcal{A}$. Consider the triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ defined as follows:

- $\mathcal{H}_\omega := \mathcal{I}^2(\mathcal{H}) \equiv \{\text{Hilbert-Schmidt operators on } \mathcal{H}\}$ with the scalar product $\langle T, S\rangle_{\mathcal{H}_\omega} := \mathrm{Tr}_{\mathcal{H}}(T^*S)$.
- For each $A \in \mathcal{A}$, let $\pi_\omega(A) : \mathcal{H}_\omega \to \mathcal{H}_\omega$ be defined by $\pi_\omega(A)T := AT$ for $T \in \mathcal{H}_\omega$.
- $\Omega_\omega := \rho^{1/2}$.

(i) Prove that Ω_ω is a unit vector in $\mathcal{H}_\omega$.

(ii) Prove that π_ω is a faithful reresentation of the C^*-algebra $\mathcal{A}$ into $\mathcal{L}(\mathcal{H}_\omega)$, i.e. that $\mathrm{Ker}(\pi_\omega) = \{0\}$.

(iii) Prove that Ω_ω is a cyclic vector for the representation π_ω, i.e. that the set $\{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$.

(iv) Prove that $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega\rangle$ for all $A \in \mathcal{A}$.

Ex. 4: Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$.

(i) Prove that the linear subspace of $\mathcal{F}_{s/a}(\mathcal{H})$ consisting of vectors $\psi = (\psi^N)_{N\in\mathbb{N}_0}$ with a finite number of nonzero components ψ_N is dense in $\mathcal{F}_{s/a}(\mathcal{H})$.

(ii) Prove that $\Gamma(A)b(A^*\varphi) = b(\varphi)\Gamma(A)$ for all $\varphi \in \mathcal{H}$, where $b(\varphi)$ are the annihilation operators on $\mathcal{F}_{s/a}(\mathcal{H})$.

(iii) Assume in addition that $A = A^*$. Prove that $\Gamma(e^{itA}) = e^{itd\Gamma(A)}$.