MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN

PROF. DR. S. BACHMANN
PROF. DR. B. PAREDES

SOSE 2017 **Mathematical Statistical Physics** 26.04.2017

Problem Sheet 1

Hand-in deadline: 03.05.2017 before 08:15 in the designated MSP box (1st floor, next to the library).

Ex. 1: For $L > 0$ let $\Lambda_L = [-L/2, L/2]^d$ be the d-dimensional cube of side length L. By identifying opposite faces we can identify Λ_L with the d-torus $(L\mathbb{T})^d = (L\mathbb{S}) \times \ldots \times (L\mathbb{S})$. Functions on the torus are viewed as functions ψ on $\mathbb{R}^d$ such that $\psi(x + k) = \psi(x)$ for all $x \in \mathbb{R}^d$ and all $k \in (L\mathbb{Z})^d$. The volume element on the d-torus is simply the restriction of d-dimensional Lebesgue measure to Λ_L. The Hilbert space $\mathcal{H} = L^2((L\mathbb{T})^d)$ is defined with respect to this measure. In particular,

$$\|\psi\|_{\mathcal{H}}^2 = \int_{\Lambda_L} |\psi(x)|^2 dx, \quad \psi \in \mathcal{H}.$$

We denote by Δ_L the Laplacian on $(L\mathbb{T})^d$. Its action on functions $\psi \in C^\infty((L\mathbb{T})^d)$ is given by $\sum_{j=1}^d \partial_j^2 \psi$.

(a) Show that $-\Delta_L$ is essentially selfadjoint on $C^\infty((L\mathbb{T})^d)$. We continue to denote the unique selfadjoint extension by $-\Delta_L$.

(b) Show that the spectrum $\sigma(-\Delta_L)$ consists only of eigenvalues $0 \leq \lambda_1^{(L)} \leq \lambda_2^{(L)} \leq \ldots$ and determine the corresponding eigenfunctions $\phi_j^{(L)}$.

(c) Prove that $-\Delta_L$ has compact resolvent. More precisely, prove that $(-\Delta_L + 1)^{-1} \in \mathcal{I}^p(\mathcal{H})$ for any $p > d/2$.[1]

(d) For any bounded continuous function $f : \mathbb{R}_+ \to \mathbb{C}$ we define

$$f(-\Delta_L) := \sum_{j=0}^{\infty} f(\lambda_j^{(L)}) |\phi_j^{(L)}\rangle\langle\phi_j^{(L)}|.$$

Show that this defines a bounded operator on $\mathcal{H}$.

(e) Show that $e^{\Delta_L} \in \mathcal{I}^1(\mathcal{H})$.

[1] Here $\mathcal{I}^p(\mathcal{H})$ is the Schatten-Von Neumann ideal consisting of compact operators A on $\mathcal{H}$ such that $\mathrm{tr}(A^*A)^{p/2} < \infty$.

(f) We identify $L^2(\Lambda_L)$ with a closed subspace of $L^2(\mathbb{R}^d)$. Prove that for any $\psi \in L^2(\mathbb{R}^d)$, we have the *strong resolvent convergence*

$$\lim_{L\to\infty} \|(-\Delta_L + 1)^{-1} P_L \psi - (-\Delta_{\mathbb{R}^d} + 1)^{-1}\psi\| = 0.$$

Here P_L is the orthogonal projection in $L^2(\mathbb{R}^d)$ onto $L^2(\Lambda_L)$, i.e. $P_L\psi = \mathbf{1}_{\Lambda_L}\psi$.

(g) Show that for any bounded continuous function $f : \mathbb{R}_+ \to \mathbb{C}$ and any $\psi \in L^2(\mathbb{R}^d)$,

$$\lim_{L\to\infty} \|f(-\Delta_L)P_L\psi - f(-\Delta_{\mathbb{R}^d})\psi\| = 0.$$

Here, $f(-\Delta_{\mathbb{R}^d})$ is defined by

$$(f(-\Delta_{\mathbb{R}^d})\psi)(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(|\xi|^2)\widehat{\psi}(\xi) d\xi$$

for $\psi \in \mathcal{S}(\mathbb{R}^d)$ and extended by continuity to all $L^2(\mathbb{R}^d)$.[2]

Jean-Claude Cuenin

[2]By Plancherel's theorem, the Fourier transformation is an isometry on $L^2(\mathbb{R}^d)$