1. Free dynamics on Fock space

We start by introducing the non-interacting dynamics on Fock spaces. Let $\mathcal{H}$ be the one particle Hilbert space and $\mathcal{F}_\pm(\mathcal{H})$ the corresponding fermionic/bosonic Fock space. The one-particle dynamics is generated by a self-adjoint operator H on $\mathcal{H}$, and it can be lifted to a non-interacting dynamics on Fock space generated by

$$\mathrm{d}\Gamma(H)\restriction_{\mathcal{H}^{\otimes n}} = H\otimes 1\cdots 1 + 1\otimes H\otimes 1\cdots 1 + \cdots + 1\cdots 1\otimes H$$

which is closeable with a self-adjoint closure. The operator $\mathrm{d}\Gamma(H)$ leaves the symmetric and antisymmetric subspaces invariant and can therefore be restricted to $\mathcal{F}_\pm(\mathcal{H})$. The tensor product structure indicates that the particles do not interact. Note that with this notation, the number operator is $N = \mathrm{d}\Gamma(1)$. Furthermore,

$$\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(H)} = \Gamma(\mathrm{e}^{-\mathrm{i}tH})$$

where

$$\Gamma(U)\restriction_{\mathcal{H}^{\otimes n}} = U\otimes\cdots\otimes U,$$

and the Heisenberg dyanmics reads $\tau_t(A) = \Gamma(\mathrm{e}^{\mathrm{i}tH})A\Gamma(\mathrm{e}^{-\mathrm{i}tH})$. Its action on creation and annihilation operators is given concretely by

$$\tau_t(b_\pm(f)) = b_\pm(\exp(\mathrm{i}tH)f),\qquad \tau_t(b_\pm^*(f)) = b_\pm^*(\exp(\mathrm{i}tH)f),$$

which is a simple strongly continuous group of Bogoliubov automorphisms. This follows from

$$\tau_t(b_\pm^*(f))\Omega = \Gamma(\mathrm{e}^{\mathrm{i}tH})(0,f,0,\cdots) = (0,\exp(\mathrm{i}tH)f,0,\cdots),$$

and

$$\begin{aligned}\|\tau_t(b_-(f)) - b_-(f)\| &= \left\|b_-\left((\mathrm{e}^{\mathrm{i}tH}-1)f\right)\right\| = \left\|(\mathrm{e}^{\mathrm{i}tH}-1)f\right\| \longrightarrow 0, &\quad \text{(fermions)}\\ \left\|\left(\tau_t(W_+(f)) - W_+(f)\right)\psi\right\| &= \left\|\left(W_+\left((\mathrm{e}^{\mathrm{i}tH}f\right) - W(f)\right)\psi\right\| \longrightarrow 0, &\quad \text{(bosons)}\end{aligned}$$

as $t\to 0$ by the strong continuity of the one-particle unitary group, and in the bosonic case the fact that the Fock representation is regular.

2. The ideal Fermi gas

We now consider a gas of non-interacting fermions, first in a finite volume $\Lambda\subset\mathbb{R}^d$, and then in the thermodynamic limit $\Lambda\to\mathbb{R}^d$ with the density $\rho_\Lambda\to\rho>0$.

Let $0<\beta<\infty$ and $\mu\in\mathbb{R}$. If

$$K_\mu := \mathrm{d}\Gamma(H-\mu 1) = \mathrm{d}\Gamma(H) - \mu N,$$

is such that $\exp(-K_\mu)$ is a trace-class operator, then the Gibbs grand canonical equilibrium state is the state over the CAR algebra $\mathcal{A}_-(\mathcal{H})$ given by

$$\omega_-^{\beta,\mu}(A) = \frac{\mathrm{Tr}_{\mathcal{F}_-(\mathcal{H})}(\exp(-\beta K_\mu)A)}{\mathrm{Tr}_{\mathcal{F}_-(\mathcal{H})}(\exp(-\beta K_\mu))} \tag{2.1}$$

Note the slight notational abuse that $A\in\mathcal{A}_-(\mathcal{H})$ on the l.h.s, while it is it Fock space representation appearing on the r.h.s. β is the inverse temperature and μ the chemical potential. We denote $z:=\exp(\beta\mu)$ and call it the activity. We have

Proposition 2.1. *$\exp(-\beta H)$ is trace-class on $\mathcal{H}$ iff $\exp(-\beta K_\mu)$ is trace-class on $\mathcal{F}_-(\mathcal{H})$ for all $\mu\in\mathbb{R}$.*

Proof. If $\exp(-\beta K_\mu)$ is trace-class then $\exp(-\beta K_\mu)\restriction_{\mathcal{H}} = z\exp(-\beta H)$ is in particular trace-class. Reciprocally, let $\{E_n\}_{n\in\mathbb{N}}$ be the eigenvalues of H in increasing order. Then

$$\begin{aligned}\mathrm{Tr}_{\mathcal{F}_-(\mathcal{H})}\mathrm{e}^{-\beta K_\mu} &= \sum_{m\geq 0} z^m \mathrm{Tr}_{\mathcal{H}_-^{(m)}}\mathrm{e}^{-\beta H^{\otimes m}} = \sum_{m\geq 0} z^m \sum_{0\leq n_1\leq\ldots\leq n_m} \mathrm{e}^{-\beta\sum_{p=1}^m E_{n_p}} = \prod_{m\geq 0}\left(1+z\mathrm{e}^{-\beta E_m}\right)\\ &\leq \prod_{m\geq 0}\exp\left(z\mathrm{e}^{-\beta E_m}\right) = \exp\left(z\mathrm{Tr}(\mathrm{e}^{-\beta H})\right),\end{aligned}$$

concluding the proof. □

Calculations in the grand canonical ensemble are easily carried out using the following pull-through formula:

$$\mathrm{e}^{-\beta K_\mu} b_-^*(f) = z b_-^*(\mathrm{e}^{-\beta H} f)\mathrm{e}^{-\beta K_\mu}. \tag{2.2}$$

In particular,

Proposition 2.2. *Assume that* $\exp(-\beta H)$ *is trace-class, and let* $\omega_-^{\beta,\mu}$ *denote the grand canonical ensemble at* $0<\beta<\infty, \mu\in\mathbb{R}$. *Then*

$$\omega_-^{\beta,\mu}(a_-^*(f)) = 0 \quad \textit{and} \quad \omega_-^{\beta,\mu}(a_-^*(f)a_-(g)) = \left\langle g, z\mathrm{e}^{-\beta H}(1+z\mathrm{e}^{-\beta H})^{-1} f\right\rangle$$

for any $f,g\in\mathcal{H}$.

Proof. By Definition (2.1) and the pull-though formula,

$$\begin{aligned}\omega_-^{\beta,\mu}(a_-^*(f)a_-(g)) &= \frac{z}{\mathrm{Tr}_{\mathcal{F}_-(\mathcal{H})}(\exp(-\beta K_\mu))}\mathrm{Tr}_{\mathcal{F}_-(\mathcal{H})}(b_-^*(\mathrm{e}^{-\beta H}f)\mathrm{e}^{-\beta K_\mu}b_-(g)) = z\omega_-^{\beta,\mu}(a_-(g)a_-^*(\mathrm{e}^{-\beta H}f))\\ &= -z\omega_-^{\beta,\mu}(a_-^*(\mathrm{e}^{-\beta H}f)a_-(g)) + z\langle g, \mathrm{e}^{-\beta H}f\rangle\end{aligned}$$

by the CAR. Hence, $\omega_-^{\beta,\mu}(a_-^*((1+z\mathrm{e}^{-\beta H})f)a_-(g)) = \langle g, z\mathrm{e}^{-\beta H}f\rangle$. The first statement follows analogously, with $\mathrm{Tr}_{\mathcal{F}_-(\mathcal{H})}(b_-^*(\mathrm{e}^{-\beta H}f)\mathrm{e}^{-\beta K_\mu}) = 0$ since K_μ preserves the particle number. □

With the same strategy, one could prove by induction that the expectation value of a product of n creation and n annihilation operators can be expressed as a polynomial in the two-point functions $\omega_-^{\beta,\mu}(a_-^*(f_i)a_-(g_j))$, namely

$$\omega_-^{\beta,\mu}(a_-^*(f_n)\cdots a_-^*(f_1)a_-(g_1)\cdots a_-(g_n)) = \det\left[\left(\left\langle g_i, z\mathrm{e}^{-\beta H}(1+z\mathrm{e}^{-\beta H})^{-1}f_j\right\rangle\right)_{i,j=1}^n\right]$$

and that the expectation value of a product with a different number of creation and annihilation operators vanish. Hence $\omega_-^{\beta,\mu}$ is a gauge-invariant quasi-free state on $\mathcal{A}_-(\mathcal{H})$.

We also note that the only property we have used is that the map $t\mapsto\tau_t^\mu(A) = \exp(-\mathrm{i}tK_\mu)A\exp(\mathrm{i}tK_\mu)$ has an analytic extension to the strip $\{z\in\mathbb{C}: 0\leq\Im z<\beta\}$ which is continuous on its closure and that the state $\omega_-^{\beta,\mu}$ has the property that

$$\omega_-^{\beta,\mu}(a_-^*(f)A) = \omega_-^{\beta,\mu}(A\tau_{\mathrm{i}\beta}^\mu(a_-^*(f)))$$

which is the so-called KMS condition at inverse temperature β. Note that this condition requires only the self-adjointness of K_μ, and no trace-class condition. In other words, the Gibbs state is the unique (τ^μ,β)-KMS state whenever $\exp(-\beta H)$ is trace-class.

We now concentrate on the special case of $H=-\Delta$ defined on $\mathcal{H}=L^2(\mathbb{R}^d)$ with domain $\mathcal{D}=H^2(\mathbb{R}^d)\equiv W^{2,2}(\mathbb{R}^d)$ and action given by

$$(Hf)(x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}|\xi|^2\hat{f}(\xi)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}\xi.$$

This H having purely absolutely continuous spectrum $\exp(-\beta H)$ cannot be trace class, but the equilibrium state corresponding to the dynamics $\tau_t(a(f)) = a(\exp(\mathrm{i}tH)f)$ can be obtained as a limit of finite volume Gibbs states. For simplicity, we consider $H_L=-\Delta$ on $L^2([-L,L]^d)$ with Dirichlet boundary conditions and denote the finite volume dynamics by $\tau_t^L(a(f)) = a(\exp(\mathrm{i}tH_L)f)$. Note that H_L has compact resolvent and that $\exp(-\beta H_L)$ is trace-class.

Theorem 2.3. *Let $\omega_{-,L}^{\beta,\mu}$ denote the Gibbs grand canonical ensemble at $0<\beta<\infty, \mu\in\mathbb{R}$ associated to H_L. For any $A\in\mathcal{A}_-(L^2([-L,L]^d))$,*

$$\lim_{L\to\infty}\omega_{-,L}^{\beta,\mu}(A) = \omega_-^{\beta,\mu}(A),$$

where $\omega_-^{\beta,\mu}$ is the gauge-invariant quasi-free state over $\mathcal{A}_-(L^2(\mathbb{R}^d))$ with two-point function

$$\omega_-^{\beta,\mu}(a_-^*(f)a_-(g)) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\frac{z\mathrm{e}^{-\beta|\xi|^2}}{1+z\mathrm{e}^{-\beta|\xi|^2}}\hat{f}(\xi)\mathrm{d}\xi.$$

Proof. Since $x\mapsto z\mathrm{e}^{-\beta x}(1+z\mathrm{e}^{-\beta x})$ is bounded function and $H_L\to H$ in the strong resolvent sense,

$$\left\langle g, z\mathrm{e}^{-\beta H_L}(1+z\mathrm{e}^{-\beta H_L})^{-1}f\right\rangle \longrightarrow \left\langle g, z\mathrm{e}^{-\beta H}(1+z\mathrm{e}^{-\beta H})^{-1}f\right\rangle$$

as $L\to\infty$, proving the convergence of $\omega_{-,L}^{\beta,\mu}(a_-^*(f)a_-(g))$ to $\omega_-^{\beta,\mu}(a_-^*(f)a_-(g))$ and thereby the weak-* convergence of $\omega_{-,L}^{\beta,\mu}$ to $\omega_-^{\beta,\mu}$. □

It is essential to note here that the limit is unique (and the limit is in fact independent on the choice of self-adjoint realisation of $-\Delta$ in finite volumes): the free Fermi gas in the infinite volume limit has a unique equilibrium state for all $0<\beta<\infty, \mu\in\mathbb{R}$. We also obtain the density of the gas as the limit

$$\rho(\beta,\mu) = \lim_{L\to\infty}(2L)^{-d}\sum_{n\geq 0}\omega_{-,L}^{\beta,\mu}(a_-^*(f_n)a_-(f_n)) = (2\pi)^{d/2}\int_{\mathbb{R}^d}\frac{z\mathrm{e}^{-\beta|\xi|^2}}{1+z\mathrm{e}^{-\beta|\xi|^2}}\mathrm{d}\xi.$$

where $(f_n)_{n\in\mathbb{N}}$ is a basis of $L^2([-L,L]^d)$. Since ξ is the the quantum mechanical momentum, it is natural to interpret $\frac{z\mathrm{e}^{-\beta|\xi|^2}}{1+z\mathrm{e}^{-\beta|\xi|^2}}$ as the momentum density distribution. Its zero-temperature limit

$$\lim_{\beta\to\infty}\frac{\mathrm{e}^{-\beta(|\xi|^2-\mu)}}{1+\mathrm{e}^{-\beta(|\xi|^2-\mu)}} = \begin{cases}1 & \text{if } |\xi|^2<\mu\\ 0 & \text{if } |\xi|^2>\mu\end{cases}$$

is called the Fermi sea.

Since $\omega_-^{\beta\mu}$ has a finite density in infinite volume, it cannot be represented on Fock space. It is however easy to check that the following Araki-Wyss representation is a GNS representation of $\mathcal{A}_-(L^2(\mathbb{R}^d))$ associated with $\omega_-^{\beta,\mu}$:

$$\mathcal{H}_\rho = \mathcal{F}_-(\mathcal{H})\otimes\mathcal{F}_-(\mathcal{H}),\qquad \Omega_\rho = \Omega\otimes\Omega,$$
$$\pi_\rho(a_-^*(f)) = b_-^*((1-\rho)^{1/2}f)\otimes 1 + (-1)^N\otimes b_-(\rho^{1/2}f),$$

where $0\leq\rho = z\exp(-\beta(-\Delta))(1+\exp(-\beta(-\Delta)))^{-1}\leq 1$ as an operator on $\mathcal{H}=L^2(\mathbb{R}^d)$. This has a natural interpretation in the case of $\rho=\rho^2$, namely at zero temperature. If $f\in\mathrm{Ker}\rho$, then $\pi_\rho(a_-^*(f))$ creates a particle upon the Fermi sea, while if $f\in\mathrm{Ran}\rho$, then $\pi_\rho(a_-^*(f))$ removes one from the Fermi sea — or in other words creates a hole.

3. The ideal Bose gas

The ideal Bose gas in finite volume is described on the bosonic Fock space $\mathcal{F}_+(\mathcal{H})$ constructed on a one-particle Hilbert space $\mathcal{H}$. As in the fermionic case, the dynamics corresponds to a group of Bogoliubov transformations defined here by

$$\tau_t(W_+(f)) = W_+(\mathrm{e}^{\mathrm{i}tH}f)$$

The Gibbs grand canonical ensemble is again defined in term of the operator K_μ, and it is well-defined whenever $\exp(-\beta K_\mu)$ is trace-class. We have:

Proposition 3.1. *Let $0<\beta<\infty$. Then $\exp(-\beta H)$ is trace-class on $\mathcal{H}$ and $H-\mu>0$ iff $\exp(-\beta K_\mu)$ is trace-class on $\mathcal{F}_+(\mathcal{H})$.*

Proof. Let $\{E_n\}_{n\in\mathbb{N}}$ be the eigenvalues of H in increasing order. Then

$$\mathrm{Tr}_{\mathcal{F}_+(\mathcal{H})}\mathrm{e}^{-\beta K_\mu}=\sum_{m\geq 0}z^m\mathrm{Tr}_{\mathcal{H}_+^{(m)}}\mathrm{e}^{-\beta H^{\otimes m}}=\sum_{m\geq 0}z^m\sum_{n_1,\ldots,n_m\geq 0}\mathrm{e}^{-\beta\sum_{p=1}^m E_{n_p}}=\prod_{k\geq 0}\sum_n \mathrm{e}^{-\beta(E_k-\mu)n}, \tag{3.1}$$

and the series converges for all k since $\beta(H-\mu)>0$, so that

$$\mathrm{Tr}_{\mathcal{F}_+(\mathcal{H})}\mathrm{e}^{-\beta K_\mu}=\prod_{k\geq 0}(1-z\mathrm{e}^{-\beta E_k})^{-1}\leq\exp(\sum_{k\geq 0}z\mathrm{e}^{-\beta E_k}(1-z\mathrm{e}^{-\beta E_k})^{-1})\leq\exp(z(1-z\mathrm{e}^{-\beta E_0})^{-1}\mathrm{Tr}(\mathrm{e}^{-\beta H}))$$

where we used that $1+x\leq\exp(x)$. Reciprocally, if $\exp(-\beta K_\mu)$ is trace-class then $\exp(-\beta K_\mu)\restriction_{\mathcal{H}}=\exp(-\beta(H-\mu))$ is in particular trace-class. But then (3.1) implies that $\beta(E_k-\mu)>0$ for all k, concluding the proof. ☐

In order to characterise explicitly the state over the CCR algabra

$$\omega_+^{\beta,\mu}(A)=\frac{\mathrm{Tr}_{\mathcal{F}_+(\mathcal{H})}(\exp(-\beta K_\mu)A)}{\mathrm{Tr}_{\mathcal{F}_+(\mathcal{H})}(\exp(-\beta K_\mu))}, \tag{3.2}$$

it can first be extended to monomials in the unbounded creation and annihilation operators (which are not in the algebra, but only in the Fock representation).

Lemma 3.2. *Let $F:=(f_1,\ldots,f_n)$ where $f_j\in\mathcal{H}$, and let $B^{\beta,\mu}(F):=b_+(f_n)\cdots b_+(f_1)\exp(-(\beta/2)K_\mu)$. Then $B^{\beta,\mu}(F)$ has a bounded closure and $B^{\beta,\mu}(F)\in\mathcal{I}_2(\mathcal{F}_+(\mathcal{H}))$.*

Proof. The condition $H-\mu>0$ implies that there is $C>0$ such that $H-\mu\cdot 1\geq C\cdot 1$ so that $K_\mu\geq C\mathcal{N}$. Since furthermore

$$\|b_+(f_n)\cdots b_+(f_1)\Psi\|\leq m^{n/2}\|\Psi\|\|f_1\|\cdots\|f_n\|$$

whenever $\Psi\in\mathcal{H}_+^{(m)}$, we have that

$$\|B^{\beta,\mu}(F)\Psi\|\leq m^{n/2}\mathrm{e}^{-(\beta/2)Cm}\|\Psi\|\|f_1\|\cdots\|f_n\|,$$

proving the boundedness of $B^{\beta,\mu}(F)$ on the dense subspace $\mathcal{F}_+^{\mathrm{fin}}(\mathcal{H})$ since $m\mapsto m^{n/2}\mathrm{e}^{-(\beta/2)Cm}$ is bounded, so that $B^{\beta,\mu}(F)$ has a bounded closure.

The creation and annihilation operators being bounded on $\mathcal{H}_+^{(m)}$, we have

$$\mathrm{Tr}_{\mathcal{H}_+^{(m)}}\left(B^{\beta,\mu}(F)^*B^{\beta,\mu}(F)\right)\leq\mathrm{Tr}_{\mathcal{H}_+^{(m)}}(\mathrm{e}^{-\beta H^{\otimes m}})(z^m m^n)\|f_1\|^2\cdots\|f_n\|^2$$

which can be summed as in the proof of Proposition 3.1. ☐

It follows that $\mathrm{Tr}(B^{\beta,\mu}(F)^*B^{\beta,\mu}(G))<\infty$ and $\mathrm{Tr}(B^{\beta,\mu}(F)B^{\beta,\mu}(G)^*)<\infty$ for any F,G as above, so that the Gibbs grand canonical state can be extended with the definition

$$\omega_+^{\beta,\mu}(b_+^*(f_1)\cdots b_+^*(f_n)b_+(g_m)\cdots b_+(g_1)):=\mathrm{Tr}_{\mathcal{F}_+(\mathcal{H})}(B^{\beta,\mu}(F)^*B^{\beta,\mu}(G)).$$

This extension is furthermore continuous since

$$\left|\mathrm{Tr}_{\mathcal{F}_+(\mathcal{H})}(B^{\beta,\mu}(F)^*B^{\beta,\mu}(G))\right|\leq C\prod_i\|f_i\|\prod_j\|g_i\|.$$

Now, the pull-through formula (2.2) remains valid in the bosonic case and yields the following:

Proposition 3.3. *Let $0<\beta<\infty,\mu\in\mathbb{R}$. Assume that $\exp(-\beta H)$ is trace-class and that $H-\mu>0$, and let $\omega_+^{\beta,\mu}$ denote the Gibbs grand canonical ensemble. Then*

$$\omega_+^{\beta,\mu}(b_+^*(f))=0\quad\textit{and}\quad\omega_+^{\beta,\mu}(b_+^*(f)b_+(g))=\left\langle g,z\mathrm{e}^{-\beta H}(1-z\mathrm{e}^{-\beta H})^{-1}f\right\rangle$$

for any $f,g\in\mathcal{H}$.

Proof. By the definition (2.1), the pull-though formula and its adjoint,

$$\begin{aligned}\omega_+^{\beta,\mu}(b_+^*(f)b_+(g)) &= \frac{1}{\mathrm{Tr}_{\mathcal{F}_+(\mathcal{H})}(\exp(-\beta K_\mu))}\mathrm{Tr}_{\mathcal{F}_+(\mathcal{H})}(b_+^*(\mathrm{e}^{-\beta(H-\mu)/2}f)\mathrm{e}^{-\beta K_\mu}b_+(\mathrm{e}^{-\beta(H-\mu)/2}g)) \\ &= \omega_+^{\beta,\mu}(b_+(\mathrm{e}^{-\beta(H-\mu)/2}g)b_+^*(\mathrm{e}^{-\beta(H-\mu)/2}f)) \\ &= \omega_+^{\beta,\mu}(b_+^*(\mathrm{e}^{-\beta(H-\mu)/2}f)b_+(\mathrm{e}^{-\beta(H-\mu)/2}g)) + \langle g, \mathrm{e}^{-\beta(H-\mu)}f\rangle\end{aligned}$$

by the CCR. This identity can be iterated n times to get

$$\omega_+^{\beta,\mu}(b_+^*(f)b_+(g)) = \omega_+^{\beta,\mu}(b_+^*(\mathrm{e}^{-n\beta(H-\mu)/2}f)b_+(\mathrm{e}^{-n\beta(H-\mu)/2}g)) + \sum_{m=1}^{n}\langle g, \mathrm{e}^{-m\beta(H-\mu)}f\rangle$$

Letting $n\to\infty$ with

$$\lim_{n\to\infty}\|\mathrm{e}^{-n\beta(H-\mu)/2}f\| = 0,$$

since $\beta(H-\mu)>0$, and using the continuity of $(f,g)\mapsto\omega_+^{\beta,\mu}(b_+^*(f)b_+(g))$, the first term vanishes while the sum of the geometric series yields the two-point function. The first statement follows as in the fermionic case. □

Here again, an iteration of the argument would prove that $\omega_+^{\beta,\mu}$ is a bosonic gauge-invariant quasi-free state, with

$$\omega_+^{\beta,\mu}(W_+(f)) = \mathrm{e}^{-\frac{1}{4}\left\langle f, \frac{1+z\mathrm{e}^{-\beta H}}{1-z\mathrm{e}^{-\beta H}}f\right\rangle}. \tag{3.3}$$

Now: the discussion of the thermodynamic limit in the case $H-\mu>0$ follows closely the fermionic case with the analogous result of a unique thermal equilibrium state in the infinite volume limit. We consider for simplicity H_L to be the Laplacian with Dirichlet boundary conditions with eigenvalues $E_{\underline{n}}(L) = (\pi^2/L^2)(n_1^2+\cdots n_d^2)$ for $\underline{n}\in(\mathbb{N})^d$ and a ground state energy $E_{\underline{1}}(L)\to 0$ as $L\to\infty$. For any $\mu<0$, namely $0<z<1$, we have that $H_L-\mu\geq-\mu>0$ uniformly for all L.

Theorem 3.4. *Let $0<\beta<\infty, \mu<0$ and let $\omega_{+,L}^{\beta,\mu}$ denote the grand canonical ensemble. For any $A\in\mathcal{A}_-(L^2([-L/2,L/2]^d))$,*

$$\lim_{L\to\infty}\omega_{+,L}^{\beta,\mu}(A) = \omega_+^{\beta,\mu}(A)$$

where $\omega_+^{\beta,\mu}$ is the gauge-invariant quasi-free state over $\mathcal{A}_+(L^2(\mathbb{R}^d))$ with two-point function

$$\omega_+^{\beta,\mu}(b_+^*(f)b_+(g)) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\frac{z\mathrm{e}^{-\beta|\xi|^2}}{1-z\mathrm{e}^{-\beta|\xi|^2}}\hat{f}(\xi)\mathrm{d}\xi. \tag{3.4}$$

Proof. It suffices to prove the convergence of the state on the Weyl operators. For this it suffices to observe that

$$0\leq\frac{1+z\mathrm{e}^{-\beta H_L}}{1-z\mathrm{e}^{-\beta H_L}}\leq\coth(\beta\mu/2),$$

which again implies the convergence of the matrix elements of $\frac{1+z\mathrm{e}^{-\beta H_L}}{1-z\mathrm{e}^{-\beta H_L}}$ to those of $\frac{1+z\mathrm{e}^{-\beta H}}{1-z\mathrm{e}^{-\beta H}}$ and thereby the weak-* convergence of $\omega_+^{\beta,\mu}(W_+(f))$, see (3.3). □

The situation is physically more interesting when the condition $H-\mu>0$ is violated: this is the phenomenon of *Bose-Einstein condensation*, one of the prime example of a phase transition. As a motivation, let us consider the density, which is in finite volume

$$\rho_L(\beta,z) = L^{-d}\sum_{\underline{n}}\omega_{+,L}^{\beta,\mu}(b_+^*(f_{\underline{n}})b_+(f_{\underline{n}})) = L^{-d}\sum_{\underline{n}}\frac{z\mathrm{e}^{-\beta E_{\underline{n}}(L)}}{1-z\mathrm{e}^{-\beta E_{\underline{n}}(L)}}, \tag{3.5}$$

where we used a basis $(f_{\underline{n}})_{\underline{n}\in\mathbb{N}^d}$ of eigenvectors of the Laplacian, corresponding to the eigenvalues $E_{\underline{n}}(L)$. Note that

$$\lim_{\mu\to 0^-}\rho_L(\beta,z)=\infty$$

at fixed (β,L), as the first term of the series diverges. The map $(0,1)\ni z\mapsto\rho_L(\beta,z)\in(0,\infty)$ being a bijection, any given density ρ can be obtained at given β,L by adjusting the chemical potential μ.

This is however not true anymore in the thermodynamic limit, where the limit $L\to\infty$ is taken first, since the density given in the above theorem

$$\rho(\beta,z)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\frac{z\mathrm{e}^{-\beta|\xi|^2}}{1-z\mathrm{e}^{-\beta|\xi|^2}}\mathrm{d}\xi,$$

which is again a monotone increasing unction of $z\in(0,1)$, has a finite limit as $z\to 1^-$. If the physical density is higher that the critical value $\rho_c(\beta):=\rho(\beta,1)$, the excess particles will all gather in the single ground state mode $\underline{n}=\underline{1}$, respectively $\xi=0$, yielding an additional δ-contribution to the density: This is the phenomenon of Bose-Einstein condensation.

Note that the above argument holds only if $d\geq 3$. Indeed at $z=1$, the integrand is of order $|\xi|^{-2}$ as $|\xi|\to 0$, so that the integral is in fact divergent at $\xi=0$ in dimensions $d=1,2$. Hence, there is no critical density and therefore also no Bose-Einstein condensation in low dimensions.

To understand this further, we first note that at fixed activity $z<1$, the single mode occupation numbers are bounded,

$$\omega_{+,L}^{\beta,\mu}(b_+^*(f_{\underline{n}})b_+(f_{\underline{n}}))=\frac{1}{z^{-1}\mathrm{e}^{\beta E_{\underline{n}}(L)}-1}\leq\frac{1}{z^{-1}-1}$$

uniformly in L. Let us now consider the particular scaling $z=z(L)=1-1/(\rho_0 L^d)$, with $0<\rho_0<\infty$ being fixed, and we temporarily consider the Laplacian with periodic boundary conditions for simplicity, for which the ground state energy is exactly $E_0(L)=0$ for all $L\in(0,\infty)$. Then

$$\mathcal{N}_{0,L}(\beta):=\omega_{+,L}^{\beta,\mu}(b_+^*(f_0)b_+(f_0))=\frac{z}{1-z}=\rho_0 L^d+o(L^d),$$

as $L\to\infty$, while if $n\neq 0$,

$$\mathcal{N}_{\underline{n},L}(\beta):=\omega_{+,L}^{\beta,\mu}(b_+^*(f_{\underline{n}})b_+(f_{\underline{n}}))=\frac{1}{z^{-1}\mathrm{e}^{\beta E_{\underline{n}}(L)}-1}\leq\frac{1}{\beta E_{\underline{n}}(L)}\leq\mathrm{const}\cdot L^2.$$

In other words, in dimension $d=3$, the ground state is the only macroscopically occupied state. It follows that for any $\varphi\in C_c^\infty(\mathbb{R}^3)$,

$$L^{-3}\sum_{\underline{n}}\mathcal{N}_{\underline{n},L}(\beta)\varphi(\underline{n})\longrightarrow(2\pi)^3\int_{\mathbb{R}^3}\mathcal{N}_s(\beta,\xi)\varphi(\xi)\mathrm{d}\xi$$

as $L\to\infty$, where

$$\mathcal{N}_s(\beta,\xi)=\overline{\mathcal{N}}(\beta,\xi)+\rho_0\delta(\xi)$$

and

$$\overline{\mathcal{N}}(\beta,\xi)=\frac{1}{(2\pi)^3}\frac{1}{\mathrm{e}^{\beta|\xi|^2}-1}.$$

Indeed, the δ-contribution arises from the ground state term in the sum; For the others, we first note that $\mathcal{N}_{\underline{n},L}(\beta)-(2\pi)^3\overline{\mathcal{N}}(\beta,\underline{n})=(1-z^{-1})\mathrm{e}^{\beta E_{\underline{n}}}\mathcal{N}_{\underline{n},L}(\beta)\overline{\mathcal{N}}(\beta,\underline{n})$. Furthermore, $\mathrm{e}^{\beta E_{\underline{n}}}\mathcal{N}_{\underline{n},L}(\beta)\leq 1+\mathcal{N}_{\underline{n},L}(\beta)\leq\mathrm{const}\cdot L^2$, so that

$$L^{-3}\sum_{n\neq 0}|\mathcal{N}_{\underline{n},L}(\beta)-(2\pi)^3\overline{\mathcal{N}}(\beta,\underline{n})|\leq\left(\mathrm{const}\cdot L^2\frac{1}{\rho_0 L^3}\right)\left(\frac{(2\pi)^3}{L^3}\sum_{n\neq 0}\overline{\mathcal{N}}(\beta,\underline{n})\right).$$

The second bracket is bounded above by $\int_{\mathbb{R}^3}\overline{\mathcal{N}}(\beta,\xi)\mathrm{d}\xi$ which, once again, is finite in three dimensions (or higher).

It follows in particular that, in the scaling limit,

$$\rho_L(\beta, z(L)) \longrightarrow \rho_s(\beta) = \overline{\rho}(\beta) + \rho_0$$

where ρ_0 denotes the condensate density and

$$\overline{\rho}(\beta) = \int_{\mathbb{R}^3} \overline{\mathcal{N}}(\beta, \xi)\mathrm{d}\xi.$$

Instead of imposing a scaling of the activity, a more natural analysis can be also be carried out at fixed density. The following proposition, in which we revert to the Dirichlet Laplacian, shows that the activity indeed converges to 1 whenever the density is larger than the critical density. For this, we note that both $z \mapsto \rho_L(\beta, z)$ and $z \mapsto \rho(\beta, z)$ are strictly increasing, so that the equation $\rho(\beta, z) = \bar{\rho}$ has a unique solution $\bar{z}$ for all $0 < \bar{\rho} \leq \rho_c(\beta) = \rho(\beta, 1)$, and $\rho_L(\beta, z) = \bar{\rho}$ has a unique solution z_L for all $0 < \bar{\rho}$.

Proposition 3.5. *Let $d \geq 3$, with $\bar{\rho} > 0$ and $0 < \beta < \infty$. For any $\bar{\rho} > 0$, let z_L be the unique solution of*

$$\rho_L(\beta, z_L) = \bar{\rho},$$

and recall that $\rho_c(\beta) := \rho(\beta, 1)$.

i. *If $\bar{\rho} \leq \rho_c(\beta)$ and $\bar{z}$ is such that $\rho(\beta, \bar{z}) = \bar{\rho}$, then $\lim_{L\to\infty} z_L = \bar{z}$*
ii. *If $\bar{\rho} > \rho_c(\beta)$, then $\lim_{L\to\infty} z_L = 1$.*

As can be expected from the discussion above, in case (ii), the surplus density $\bar{\rho} - \rho_c(\beta)$ condensates into the ground state, and indeed

$$\lim_{L\to\infty} L^{-d} \frac{z_L \mathrm{e}^{-\beta E_{\underline{1}}(L)}}{1 - z_L \mathrm{e}^{-\beta E_{\underline{1}}(L)}} = \bar{\rho} - \rho_c(\beta)$$

where $E_{\underline{1}}(L)$ is the ground state energy of the Dirichlet Laplacian.

Proof. (i) From the convexity of $z \mapsto \rho_L(\beta, z)$, we have that

$$\frac{\partial \rho_L}{\partial z}(\beta, z_2) \leq \frac{\rho_L(\beta, z_1) - \rho_L(\beta, z_2)}{z_1 - z_2} \leq \frac{\partial \rho_L}{\partial z}(\beta, z_1)$$

whenever $z_2 < z_1$. Moreover, the explicit expression (3.5) implies that

$$\frac{\rho_L(\beta, z)}{z} \leq \frac{\partial \rho_L}{\partial z}(\beta, z)$$

so that

$$\frac{\rho_L(\beta, z_2)}{z_2} \leq \frac{\rho_L(\beta, z_1) - \rho_L(\beta, z_2)}{z_1 - z_2}. \tag{3.6}$$

Noting that $\rho_L(\beta, z) \leq \rho(\beta, z)$ by a Riemann approximation argument, and that both are increasing functions of z, we have that $z_L \geq \bar{z}$. By (3.6),

$$0 \leq z_L - \bar{z} \leq \frac{\bar{z}(\bar{\rho} - \rho_L(\beta, \bar{z}))}{\rho_L(\beta, \bar{z}))}$$

proving that $\lim_{L\to\infty} z_L = \bar{z}$.
(ii) Assume that $z_L \leq 1$. Then $\rho_c(\beta) < \bar{\rho} = \rho_L(\beta, z_L) \leq \rho(\beta, z_L) \leq \rho_c(\beta)$, which is a contradiction. Hence $z_L > 1$. But $z_L < \exp(\beta E_{\underline{1}}(L))$, which converges to 1, so that $\lim_{L\to\infty} z_L = 1$. □

With a little more effort, one can prove the following theorem, completely characterising the Gibbs grand canonical equilibrium states in the thermodynamic limit.

Theorem 3.6. *Let $d \geq 3$, with $\bar{\rho} > 0$ and $0 < \beta < \infty$. Let $\omega_{+,L}^{\beta,\mu_L}$ be the Gibbs grand canonical equilibrium state with μ_L chosen so that $\rho_L(\beta, z_L) = \bar{\rho}$. Then the weak-* limit $\lim_{L\to\infty} \omega_{+,L}^{\beta,\mu_L} = \omega_+^\beta$ exists and is a gauge-invariant quasi-free state. Furthermore,*

i. *If* $\bar\rho \le \rho_c(\beta)$ *then the two-point function of* ω_+^β *is given by (3.4) where* z *is the solution of* $\rho(\beta, z) = \bar\rho$

ii. *If* $\bar\rho > \rho_c(\beta)$, *then the two-point function of* ω_+^β *is given by*

$$\omega_+^\beta(b_+^*(f)b_+(g)) = (4\pi)^d(\bar\rho - \rho_c(\beta))\overline{\hat g(0)}\hat f(0) + \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\overline{\hat g(\xi)}\frac{\mathrm{e}^{-\beta|\xi|^2}}{1-\mathrm{e}^{-\beta|\xi|^2}}\hat f(\xi)\mathrm{d}\xi.$$

Note that by rescaling ξ, one obtains $\rho_c(\beta) = \mathrm{const}\cdot\beta^{-d/2}$ showing that the critical density is a strictly increasing, convex function of the temperature. Hence the condensation regime is reached at fixed density $\bar\rho$ by lowering the temperature below a critical value. In other words, Bose-Einstein condensation occurs in a low temperature, high density regime.

Summarising the above discussion, the ‘normal regime’ is characterised by a unique equilibrium state for any β, μ given by Theorem 3.4. In the condensation regime, there are infinitely many equilibrium states, all having the same temperature and chemical potential, and they are parametrised by the physical density $\bar\rho \in [\rho_c(\beta), \infty)$. This ‘bifurcation’ from a unique to many equilibrium states is a characteristic property of a thermal phase transition.

A proof of the existence of Bose-Einstein condensation for an *interacting* Bose gas is still missing. However, progress has been made in the so-called Gross-Pitaevskii limit, a regime of very few but very strong interaction (Lieb-Seiringer-Yngvason, Phys. Rev. A 61, 043602, 2000), or in a toy model of spins on a lattice where the phenomenon of gauge symmetry breaking is clarified (Lieb-Seiringer-Yngvason, Rep. Math. Phys. 59(3), 389, 2007)