Chapter 2

C*-algebras, states and representations

2.1 C*-algebras

Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$. $\mathcal{A}$ is a normed algebra if there is a norm $\mathcal{A} \ni x \mapsto \|x\| \in \mathbb{R}_+$ such that $\|xy\| \leq \|x\|\|y\|$. A complete normed algebra is a Banach algebra. A mapping $x \mapsto x^*$ of $\mathcal{A}$ into itself is an involution if

$$
\begin{aligned}
&(x^*)^* = x; \\
&(x+y)^* = x^* + y^*; \\
&(xy)^* = y^* x^*; \\
&(\lambda x)^* = \overline{\lambda} x^*.
\end{aligned}
$$

An algebra with an involution is a *-algebra.

Definition 1. *A Banach *-algebra $\mathcal{A}$ is called a* C*-algebra *if*

$$\|x^* x\| = \|x\|^2, \qquad x \in \mathcal{A}.$$

Proposition 1. *Let $\mathcal{A}$ be a C*-algebra.*

1. $\|x^*\| = \|x\|$;
2. *If $\mathcal{A}$ does not have an identity, let $\widetilde{\mathcal{A}}$ be the algebra obtained from $\mathcal{A}$ by adjoining an identity* 1. *Then $\widetilde{\mathcal{A}}$ is a C*-algebra with norm $\|\cdot\|$ defined by*

$$\|\lambda 1 + x\| = \sup_{y \neq 0} \frac{\|\lambda y + xy\|}{\|y\|}, \qquad \lambda \in \mathbb{C}.$$

Proof. Exercise. □

In the following, $\mathcal{A}$ will always denote a C*-algebras with an identity if not specified otherwise.

Definition 2. *The* spectrum $\mathrm{Sp}(x)$ *of $x \in \mathcal{A}$ is the set*

$$\mathrm{Sp}(x) := \{\lambda \in \mathbb{C} : x - \lambda 1 \textit{ is not invertible in } \mathcal{A}\}.$$

If $|\lambda| > \|x\|$, then the series $\lambda^{-1}\sum_{n\in\mathbb{N}}(x/\lambda)^n$ is norm convergent and sums to $(\lambda 1 - x)^{-1}$. Hence, $\mathrm{Sp}(x) \subset B_{\|x\|}(0)$. Assume now that $x \in \mathcal{A}$ is a self-adjoint element and that $a + \mathrm{i}b \in \mathrm{Sp}(x)$, $a, b \in \mathbb{R}$. Then $a+\mathrm{i}(b+t) \in \mathrm{Sp}(x+\mathrm{i}t1)$. Since $\|x+\mathrm{i}t1\|^2 = \|x+\mathrm{i}t1\|\|x-\mathrm{i}t1\| = \|x^2+t^2 1\| \leq \|x\|^2 + t^2$, and by the remark above, $|a+\mathrm{i}(b+t)|^2 \leq \|x\|^2 + t^2$, and further $2bt \leq \|x\| - a^2 - b^2$ for all $t \in \mathbb{R}$, so that $b = 0$. For any polynomial P over $\mathbb{C}$, $P(\mu) - \lambda = A\prod_{i=1}^n(\mu - z_i)$, and $P(x) - \lambda 1 = A\prod_{i=1}^n(x - z_i) \in \mathcal{A}$ for any $x \in \mathcal{A}$. Hence, $\lambda \in \mathrm{Sp}(P(x))$ iff $z_j \in \mathrm{Sp}(x)$ for a $1 \leq j \leq n$. Since $P(z_j) = \lambda$, we have that $\lambda \in \mathrm{Sp}(P(x))$ iff $\lambda \in P(\mathrm{Sp}(x))$. We have proved

Proposition 2. *Let $x \in \mathcal{A}$.*

1. $\mathrm{Sp}(x) \subset B_{\|x\|}(0)$*;*
2. *if $x = x^*$, then* $\mathrm{Sp}(x) \subset [-\|x\|, \|x\|]$*;*
3. *if $xx^* = x^*x$, i.e. x is normal, then $\|x\| = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(x)\}$;*
4. *for any polynomial P,* $\mathrm{Sp}(P(x)) = P(\mathrm{Sp}(x))$*;*

The proof of 3. is left as an exercise. Note that the condition holds in particular for $x = x^*$

An element $x \in \mathcal{A}$ is positive if it is self-adjoint and $\mathrm{Sp}(x) \subset \mathbb{R}_+$.

Proposition 3. *Let $x \in \mathcal{A}$, $x \neq 0$. The following are equivalent:*

1. *x is positive;*
2. *there is a self-adjoint $z \in \mathcal{A}$ such that $x = z^2$;*
3. *there is $y \in \mathcal{A}$ such that $x = y^*y$;*

Proof. (3) $\Rightarrow$ (2) by choosing $y = z$. (3) $\Rightarrow$ (1) since z^2 is self-adjoint and since, by Proposition 2, $\mathrm{Sp}(z^2) \subset [0, \|z\|^2]$. To show (1) $\Rightarrow$ (3), we note that[1] for any $\mu > 0$,

$$\mu = \left[\frac{1}{\pi}\int_0^\infty \sqrt{\lambda}\left(\frac{1}{\lambda} - \frac{1}{\lambda+\mu}\right)d\lambda\right]^2 \tag{2.1}$$

Since x is positive, $(x+\lambda 1)$ is invertible for all $\lambda > 0$ so that $z := \pi\int_0^\infty \sqrt{\lambda}\left(\lambda^{-1} - (x+\lambda 1)^{-1}\right)d\lambda$ is well defined as a norm convergent integral, and $x = z^2$. Using again (2.1) with $\mu = 1$, we have that

$$\|x\|^{1/2}1 - z = \frac{\|x\|^{1/2}}{\pi}\int_0^\infty \frac{\sqrt{\lambda}}{\lambda+1}(\hat{x}+\lambda 1)^{-1}(\hat{x}-1)d\lambda, \qquad \hat{x} = x\|x\|^{-1}$$

But $\hat{x}$ positive implies $\mathrm{Sp}(\hat{x}) \subset [0,1]$, hence $\mathrm{Sp}(1-\hat{x}) \subset [0,1]$ and $\|1-\hat{x}\| \leq 1$. Moreover, since $\mathrm{Sp}((\hat{x}+\lambda 1)^{-1}) = (\mathrm{Sp}(\hat{x}+\lambda 1))^{-1} \subset [(1+\lambda)^{-1}, \lambda^{-1}]$, we have $\|(\hat{x}+\lambda 1)^{-1}\| \leq \lambda^{-1}$ for $\lambda > 0$. Hence, $\|1 - z\|x\|^{-1/2}\| \leq 1$ so that $\mathrm{Sp}(z) \subset [0, 2\|x\|^{1/2}]$ and finally z is positive.

It remains to prove (2) $\Rightarrow$ (1). Since $x = y^*y$ is self-adjoint, x^2 is positive and we denote by $|x|$ its positive square root defined by the integral above. Then $x_\pm := (|x| \pm x)/2$ is positive and $x_-x_+ = x_+x_- = 0$. Decomposing $yx_- = s + \mathrm{i}t$, with self-adjoint s, t, we have $(yx_-)^*(yx_-) + (yx_-)(yx_-)^* = 2(s^2+t^2) \geq 0$. But $-(yx_-)^*(yx_-) = -x_-(-x_-+x_+)x_- = x_-^3$ is positive, so that $(yx_-)(yx_-)^*$ is positive. On the other hand, $\mathrm{Sp}((yx_-)(yx_-)^*) \cup \{0\} = \mathrm{Sp}((yx_-)^*(yx_-)) \cup \{0\} \subset \mathbb{R}_-$, hence $(yx_-)^*(yx_-) = 0$ so that $x_- = 0$, and finally $x = x_+$ is positive. □

[1] Convergence follows from the asymptotics $O(s^{-1/2})$ as $s \to 0$ and $O(s^{-3/2})$ as $s \to \infty$, while the change of variables $\lambda = \mu\xi$ yields immediately that the integral is $\sqrt{\mu}$, up to a constant.

Definition 3. *A* *-morphism *between two *-algebras $\mathcal{A}$ and $\mathcal{B}$ is a linear map $\pi : \mathcal{A} \to \mathcal{B}$ such that $\pi(A_1A_2) = \pi(A_1)\pi(A_2)$ and $\pi(A^*) = \pi(A)^*$, for all $A, A_1, A_2 \in \mathcal{A}$. It is called a* *-isomorphism *if it is bijective. A *-isomorphism $\mathcal{A} \to \mathcal{A}$ is an* automorphism.

Proposition 4. *Let $\mathcal{A}, \mathcal{B}$ be two C*-algebras and $\pi : \mathcal{A} \to \mathcal{B}$ a *-morphism. Then $\|\pi(x)\|_{\mathcal{B}} \leq \|x\|_{\mathcal{A}}$, and the range $\{\pi(A) : A \in \mathcal{A}\}$ is a *-subalgebra of $\mathcal{B}$.*

Proof. If x is self-adjoint, so is $\pi(x)$ and $\|\pi(x)\| = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(\pi(x))\}$. If $x-\lambda 1$ is invertible, then $1 = \pi((x-\lambda 1)^{-1}(x-\lambda 1)) = \pi((x-\lambda 1)^{-1})\pi(x-\lambda 1)$ so that $\pi(x)-\lambda 1$ is invertible, whence $\mathrm{Sp}(\pi(x)) \subset \mathrm{Sp}(x)$, we have that $\|\pi(x)\| \leq \sup\{|\lambda| : \lambda \in \mathrm{Sp}(x)\} = \|x\|$. The general case follows from $\|\pi(x)\|^2 = \|\pi(x^*x)\| \leq \|x^*x\| = \|x\|^2$. □

Let Γ be a locally compact Hausdorff space, and let $C_0(\Gamma)$ be the algebra, under pointwise multiplication, of all complex valued continuous functions that vanish at infinity.

Theorem 5. *If $\mathcal{A}$ is a commutative C*-algebra, then there is a locally compact Hausdorff space Γ such that $\mathcal{A}$ is *-isomorphic to $C_0(\Gamma)$.*

Proof. See Robert's lectures. □

In classical mechanics, the space Γ is usually referred to as the phase space.

If $\mathcal{A}$ is a commutative C*-algebra with an identity, then $\mathcal{A}$ is isomorphic to $C(K)$, the algebra of continuous functions on a compact Hausdorff space K.

Let $\mathcal{U}$ be a *-subalgebra of $\mathcal{L}(\mathcal{H})$. The commutant $\mathcal{U}'$ is the subset of $\mathcal{L}(\mathcal{H})$ of operators that commute with every element of $\mathcal{U}$, and so forth with $\mathcal{U}'' := (\mathcal{U}')'$. In particular, $\mathcal{U} \subset \mathcal{U}''$, and further $\mathcal{U}' = \mathcal{U}'''$.

Definition 4. *A* von Neumann algebra *or W*-algebra on $\mathcal{H}$ is a *-subalgebra $\mathcal{U}$ of $\mathcal{L}(\mathcal{H})$ such that $\mathcal{U}'' = \mathcal{U}$. Its* center *is $\mathcal{Z}(\mathcal{U}) := \mathcal{U} \cap \mathcal{U}'$, and $\mathcal{U}$ is a* factor *if $\mathcal{Z}(\mathcal{U}) = \mathbb{C} \cdot 1$.*

Theorem 6. *Let $\mathcal{U}$ be a *-subalgebra of $\mathcal{L}(\mathcal{H})$ such that $\mathcal{U}\mathcal{H} = \mathcal{H}$. Then $\mathcal{U}$ is a von Neumann algebra iff $\mathcal{U}$ is weakly closed.*

Note that $\mathcal{U}\mathcal{H} = \mathcal{H}$ is automatically satisfied if $1 \in \mathcal{U}$. Furthermore, for any *-subalgebra $\mathcal{U}$ of $\mathcal{L}(\mathcal{H})$, let $\overline{\mathcal{U}}$ be its weak closure for which $\overline{\mathcal{U}}'' = \overline{\mathcal{U}}$ by the theorem. Since $\mathcal{U} \subset \overline{\mathcal{U}}$, we have $\overline{\mathcal{U}}' \subset \mathcal{U}'$. Furthermore, if $x \in \mathcal{U}'$, and $y \in \overline{\mathcal{U}}$, with $\mathcal{U} \ni y_n \rightharpoonup y$, then x commutes with y_n for all n and hence with y, so that $\mathcal{U}' \subset \overline{\mathcal{U}}'$. It follows that $\overline{\mathcal{U}}' = \mathcal{U}'$, whence $\overline{\mathcal{U}}'' = \mathcal{U}''$ and so:

Corollary 7. *$\mathcal{U}$ is weakly dense in $\mathcal{U}''$, namely $\overline{\mathcal{U}} = \mathcal{U}''$.*

2.2 Representations and states

Definition 5. *Let $\mathcal{A}$ be a C*-algebra and $\mathcal{H}$ a Hilbert space. A* representation *of $\mathcal{A}$ in $\mathcal{H}$ is a *-morphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$. Moreover,*

1. *Two representations π, π' in $\mathcal{H}, \mathcal{H}'$ are* equivalent *if there is a unitary map $U : \mathcal{H} \to \mathcal{H}'$ such that $U\pi(x) = \pi'(x)U$;*
2. *A representation π is* topologically irreducible *if the only closed subspaces that are invariant under $\pi(\mathcal{A})$ are $\{0\}$ and $\mathcal{H}$;*
3. *A representation π is* faithful *if it is an isomorphism, namely* $\mathrm{Ker}\pi = \{0\}$.

Note that in general $\|\pi(x)\| \leq \|x\|$, with equality if and only if π is faithful. One can further show that for any C*-algebra, there exists a faithful representation.

If $(\mathcal{H}, \pi)$ is a representation of $\mathcal{A}$ and $n \in \mathbb{N}$, then $n\pi(A)(\oplus_{i=1}^n \psi_i) := \oplus_{i=1}^n \pi(A)\psi_i$ defines a representation $n\pi$ on $\oplus_{i=1}^n \mathcal{H}$.

Proposition 8. *Let π be a representation of $\mathcal{A}$ in $\mathcal{H}$. T.f.a.e*

1. *π is topologically irreducible;*
2. *$\pi(\mathcal{A})' := \{B \in \mathcal{L}(\mathcal{H}) : [B, \pi(x)] = 0,$ for all $x \in \mathcal{A}\} = \mathbb{C} \cdot 1$;*
3. *Any $\xi \in \mathcal{H}$, $\xi \neq 0$ is cyclic: $\overline{\pi(x)\xi} = \mathcal{H}$, or $\pi = 0$.*

Proof. (1) $\Rightarrow$ (3) : If $\pi(\mathcal{A})\xi$ is not dense, then $\pi(\mathcal{A})\xi = \{0\}$. It follows that $\mathbb{C}\xi$ is an invariant subspace, and hence $\mathcal{H} = \mathbb{C}\xi$ and $\pi = 0$
(3) $\Rightarrow$ (1) : Let $\mathcal{K} \neq \{0\}$ be a closed invariant subspace. For any $\xi \in \mathcal{K}$, $\pi(\mathcal{A})\xi \subset \mathcal{K}$ and since ξ is cyclic, $\pi(\mathcal{A})\xi$ is dense in $\mathcal{H}$
(2) $\Rightarrow$ (1) : Let $\mathcal{K} \neq \{0\}$ be a closed invariant subspace, and let $P_\mathcal{K}$ be the orthogonal projection on $\mathcal{K}$. Then $P_\mathcal{K} \in \pi(\mathcal{A})'$, since for $\xi \in \mathcal{K}, \eta \in \mathcal{K}^\perp$, $\langle \xi, \pi(x)\eta\rangle = \langle \pi(x^*)\xi, \eta\rangle = 0$ so that $\pi(x)\eta \in \mathcal{K}^\perp$ for any $x \in \mathcal{A}$. Hence $P_\mathcal{K} = 0$ or $P_\mathcal{K} = 1$, i.e. $\mathcal{K} = \{0\}$ or $\mathcal{K} = \mathcal{H}$.
(1) $\Rightarrow$ (2) : Let $c \in \pi(\mathcal{A})'$ be self-adjoint. Then all spectral projectors of c belong to $\pi(\mathcal{A})'$, so that they are all either 0 or 1 by (1), and c is a scalar. If c is not self-adjoint, apply the above to $c \pm c^*$. □

A triple $(\mathcal{H}, \pi, \xi)$ where ξ is a cyclic vector is called a cyclic representation.

Recall that $\mathcal{A}^* := \{\omega : \mathcal{A} \to \mathbb{C} : \omega$ is linear and bounded$\}$. For any $\xi \in \mathcal{H}$, the map $\mathcal{A} \ni x \mapsto \langle \xi, \pi(x)\xi\rangle$ is an element of $\mathcal{A}^*$ since $|\langle \xi, \pi(x)\xi\rangle| \leq \|\xi\|_\mathcal{H}^2 \|x\|_\mathcal{A}$ and it is positive: If x is positive, then $x = y^*y$ and $\langle \xi, \pi(x)\xi\rangle = \|\pi(y)\xi\|_\mathcal{H}^2 \geq 0$. We shall denote it $\omega_{\pi,\xi}$. If $0 \leq T \leq 1$ is a self-adjoint operator in $\mathcal{H}$ and $T \in \pi(\mathcal{A})'$, then the form $x \mapsto \omega_{\pi,T\xi}$ is positive, and $\omega_{\pi,T\xi}(y^*y) = \|\pi(y)T\xi\|^2 = \|T\pi(y)\xi\|^2 \leq \|\pi(y)\xi\|^2 = \omega_{\pi,\xi}(y^*y)$, so that $\omega_{\pi,T\xi} \leq \omega_{\pi,\xi}$.

Lemma 9. *Let ω be a positive linear functional on $\mathcal{A}$. Then*

$$\omega(x^*y) = \overline{\omega(y^*x)}, \qquad |\omega(x^*y)|^2 \leq \omega(x^*x)\omega(y^*y).$$

Proof. This follows from the positivity of the quadratic form $\lambda \mapsto \omega((\lambda x + y)^*(\lambda x + y)) \geq 0$. □

In fact, any positive linear form ν bounded above by $\omega_{\pi,\xi}$ is of the form above. Indeed,

$$|\nu(x^*y)|^2 \leq \nu(x^*x)\nu(y^*y) \leq \omega_{\pi,\xi}(x^*x)\omega_{\pi,\xi}(y^*y) \leq \|\pi(x)\xi\|^2 \|\pi(y)\xi\|^2$$

so that $\pi(x)\xi \times \pi(y)\xi \mapsto \nu(x^*y)$ is a densely defined, bounded, symmetric linear form on $\mathcal{H} \times \mathcal{H}$. By Riesz representation theorem, there exists a unique bounded operator T such that $\nu(x^*y) = \langle \pi(x)\xi, T\pi(y)\xi\rangle$, and $0 \leq T \leq 1$. Moreover,

$$\langle \pi(x)\xi, T\pi(z)\pi(y)\xi\rangle = \nu(x^*zy) = \nu((z^*x)^*y) = \langle \pi(x)\xi, \pi(z)T\pi(y)\xi\rangle,$$

so that $T \in (\pi(\mathcal{A}))'$.

Definition 6. *A* state *ω on a C*-algebra $\mathcal{A}$ is a positive element of $\mathcal{A}^*$ such that*

$$\|\omega\| = \sup_{x \in \mathcal{A}} \frac{\omega(x)}{\|x\|} = 1.$$

A state ω is called

- pure *if the only positive linear functionals majorised by* ω *are* $\lambda\omega$, $0 \leq \lambda \leq 1$,
- faithful *if* $\omega(x^*x) = 0$ *implies* $x = 0$.

If ω is normalised and $\mathcal{A}$ has an identity, then $\omega(1) = 1$. Reciprocally, $|\omega(x)|^2 \leq \omega(1)\omega(x^*x)$. Since $\|x^*x\|1 - x^*x \geq 0$, we further have $|\omega(x)|^2 \leq \|x^*x\|\omega(1)^2$, i.e. $\|\omega\| \leq \omega(1)$, which proves:

Proposition 10. *If* $\mathcal{A}$ *has an identity, and* ω *is a positive linear form on* $\mathcal{A}$, *then* $\|\omega\| = 1$ *if and only if* $\omega(1) = 1$.

By Corollary 7, $\pi(\mathcal{A})''$ is a von Neumann algebra for any state ω. ω is called a factor state if $\pi(\mathcal{A})''$ is a factor, i.e. if $\pi(\mathcal{A})' \cap \pi(\mathcal{A})'' = \mathbb{C} \cdot 1$.

We shall denote $\mathcal{E}(\mathcal{A})$ the set of states over $\mathcal{A}$ and $\mathcal{P}(\mathcal{A})$ the set of pure states.

Proposition 11. $\mathcal{E}(\mathcal{A})$ *is a convex set, and it is weakly-* compact iff* $\mathcal{A}$ *has an identity. In that case,* $\omega \in \mathcal{P}(\mathcal{A})$ *iff it is an extremal point of* $\mathcal{E}(\mathcal{A})$.

Proof. We only prove the second part, the first part being is a version of the Banach-Alaoglu theorem. Let $\omega \in \mathcal{P}(\mathcal{A})$. Assume that $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$. Then $\omega \geq \lambda\omega_1$, hence $\lambda\omega_1 = \mu_1\omega$, $0 \leq \mu_2 \leq 1$ and similarly for ω_2. Hence $\omega = (\mu_1 + \mu_2)\omega$ and ω is extremal. Reciprocally, assume that ω is not pure, in which case there is a linear functional $\tilde{\nu}_1 \neq \tilde{\lambda}\omega$ such that $\omega \geq \tilde{\nu}_1$. In particular, $\lambda := \tilde{\nu}_1(1) \leq \omega(1) = 1$. Since $\nu_1 := \lambda^{-1}\tilde{\nu}_1$ is a state, $\nu_2 := (\omega - \lambda\nu_1)/(1-\lambda)$ defines a state, and $\omega = \lambda\nu_1 + (1-\lambda)\nu_2$. Hence ω is not extremal. □

In particular, if $\{\omega_i\}_{i\in I}$ is an arbitrary infinite family of states, then there exists at least one weak-* accumulation point. Note that $\omega_n \rightharpoonup \omega$ in the weak-* topology if $\omega_n(x) \to \omega(x)$ for all $x \in \mathcal{A}$. In fact, it is defined as the weakest topology in which this holds, namely in which the map $x : \omega \mapsto \omega(x)$ are continuous.

Theorem 12. *Let* $\mathcal{A}$ *be a C*-algebra and* $\omega \in \mathcal{E}(\mathcal{A})$. *Then there exists a cyclic representation* $(\mathcal{H}, \pi, \Omega)$ *such that*

$$\omega(x) = \langle \Omega, \pi(x)\Omega \rangle$$

for all $x \in \mathcal{A}$. *Such a representation is unique up to unitary isomorphism.*

Proof. We consider only the case where $\mathcal{A}$ has an identity. Let $\mathcal{N} := \{a \in \mathcal{A} : \omega(a^*a) = 0\}$. Since, by Lemma 9, $0 \leq \omega(a^*x^*xa) \leq \omega(a^*a)\|x\|^2 = 0$, we have $a \in \mathcal{N}, x \in \mathcal{A}$ implies $xa \in \mathcal{N}$ is a left ideal. On $h := \mathcal{A} \setminus \mathcal{N}$, we denote ψ_x the equivalence class of $x \in \mathcal{A}$, and the bilinear form $(\psi_x, \psi_y) \mapsto \omega(x^*y)$ is positive and well-defined, since $\omega((x+a)^*, y+b) = \omega(x^*y) + \omega(a^*y) + \omega(x^*b) + \omega(a^*b) = \omega(x^*y)$ for any $x, y \in \mathcal{A}$; $a, b \in \mathcal{N}$. Let $\mathcal{H}$ be the Hilbert space completion of h. For any $\psi_x \in h$, let $\pi(y)\psi_x := \psi_{yx}$. The map $\pi : \mathcal{A} \to \mathcal{L}(h)$ is linear and bounded since $\|\pi(y)\psi_x\|^2 = \langle \psi_{yx}, \psi_{yx} \rangle = \omega(x^*y^*yx) \leq \|y\|^2\|x\|^2$ and thus has a bounded closure. It is a *-homomorphism since

$$\langle \psi_y, \pi(z^*)\psi_x \rangle = \langle \psi_y, \psi_{z^*x} \rangle = \omega(y^*z^*x) = \langle \psi_{zy}, \psi_x \rangle = \langle \pi(z)\psi_y, \psi_x \rangle$$

and $\pi(xy)\psi_z = \psi_{xyz} = \pi(x)\pi(y)\psi_z$ and defines a representation of $\mathcal{A}$ in $\mathcal{H}$. Moreover, $\langle \psi_1, \pi(x)\psi_1 \rangle = \langle \psi_1, \psi_x \rangle = \omega(x)$, so that $\Omega = \psi_1$. Cyclicity follows from $\{\pi(x)\Omega : x \in \mathcal{A}\} = \{\psi_x : x \in \mathcal{A}\}$, which is the dense set of equivalence classes by construction. Finally, let $(\mathcal{H}', \pi', \Omega')$ be another such representation. Then the map $U : \mathcal{H} \to \mathcal{H}'$ defined by $\pi'(x)\Omega' = U\pi(x)\Omega$ is a densely defined isometry, since

$$\langle \pi(y)\Omega, \pi(x)\Omega \rangle_{\mathcal{H}} = \omega(y^*x) = \langle \pi'(y)\Omega', \pi'(x)\Omega' \rangle_{\mathcal{H}'} = \langle U\pi(y)\Omega, U\pi(x)\Omega \rangle_{\mathcal{H}'},$$

and hence extends to a unitary operator. □

Corollary 13. *Let $\mathcal{A}$ be a C*-algebra and α a *-automorphism. If $\omega \in \mathcal{E}(\mathcal{A})$ is α-invariant, $\omega(\alpha(x)) = \omega(x)$ for all $x \in \mathcal{A}$, then there is a unique unitary operator U on the GNS Hilbert space $\mathcal{H}$ such that, for all $x \in \mathcal{A}$,*

$$U\pi(x) = \pi(\alpha(x))U, \quad \text{and} \quad U\Omega = \Omega.$$

One says that α is unitarily implementable in the GNS representation.

Proof. The corollary follows from the uniqueness part of Theorem 12 applied to $(\mathcal{H}, \pi \circ \alpha, \Omega)$, since $\langle \Omega, \pi(x)\Omega \rangle = \omega(x) = \omega(\alpha(x)) = \langle \Omega, \pi \circ \alpha(x)\Omega \rangle$. □

Proposition 14. *Let $\mathcal{A}$ be a C*-algebra, $\omega \in \mathcal{E}(\mathcal{A})$ and $(\mathcal{H}, \pi, \Omega)$ the associated representation. Then π is irreducible and $\pi \neq 0$ iff ω is a pure state.*

Proof. Let ν be majorised by $\omega = \omega_{\pi,\Omega}$. There is a $0 \leq T \leq 1$ such that $\nu(x^* y) = \langle \pi(x)\xi, T\pi(y)\xi \rangle$ with $T \in (\pi(\mathcal{A}))'$. If π is irreducible, then $T = \sqrt{\lambda} \cdot 1$ so that $\nu = \lambda\omega$, $0 \leq \lambda \leq 1$ and ω is pure. Reciprocally, if ν is not a multiple of ω, then T is not a multiple of the identity, so that $(\mathcal{H}, \pi)$ is not irreducible. □

Definition 7. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}$. A state ω is π-*normal *if there exists a density matrix ρ_ω in $\mathcal{H}$ such that $\omega(A) = \mathrm{Tr}(\rho_\omega \pi(A))$. Two representations $(\mathcal{H}_1, \pi_1), (\mathcal{H}_2, \pi_2)$ are* quasi-equivalent *if every π_1-normal state is π_2-normal and conversely.*

Further, two states ω_1, ω_2 are said to be quasi-equivalent if their GNS representations are quasi-equivalent. These correspond to thermodynamically equivalent states.

2.3 Examples: Quantum spin systems, the CCR and CAR algebras

2.3.1 Quantum spin systems

Let Γ be a countable set. Denote $\Lambda \Subset \Gamma$ the finite sets of Γ and $\mathcal{F}(\Gamma)$ the set of finite subsets. For each $x \in \mathcal{H}$, let $\mathcal{H}_x$ be a finite dimensional Hilbert space, and assume that $\sup_{x\in\Gamma} \dim(\mathcal{H}_x) < \infty$. The Hilbert space of $\Lambda \Subset \Gamma$ is given by $\mathcal{H}_\Lambda := \otimes_{x\in\Lambda}\mathcal{H}_x$. The associated algebra of local observables is

$$\mathcal{A}_\Lambda := \mathcal{L}(\mathcal{H}_\Lambda) \simeq \otimes_{x\in\Gamma}\mathcal{L}(\mathcal{H}_x).$$

Inclusion defines a partial order on $\mathcal{F}(\Gamma)$, which induces the following imbedding:

$$\Lambda \subset \Lambda' \quad \Longrightarrow \quad \mathcal{A}_\Lambda \subset \mathcal{A}_{\Lambda'}$$

where $x \in \mathcal{A}_\Lambda$ is identified with $x \otimes 1_{\Lambda'\setminus\Lambda} \in \mathcal{A}_{\Lambda'}$. Note that $\Lambda \cap \Lambda' = \emptyset$ implies $xy = x \otimes y = yx$ for all $x \in \mathcal{A}_\Lambda, y \in \mathcal{A}_{\Lambda'}$. Finally, the algebra of quasi-local observables is given by

$$\mathcal{A} := \overline{\bigcup_{\Lambda\in\mathcal{F}(\Gamma)} \mathcal{A}_\Lambda}^{\|\cdot\|} \equiv \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}$$

and it is a C*-algebra. Note that $\mathcal{A}$ has an identity. In other words, $\mathcal{A}$ is obtained as a limit of finite-dimensional matrix algebras, which is referred to as a uniformly hyperfine algebra (UHF). From the physical point of view, a finite dimensional Hilbert space is the state space of a physical system with a finite number of degrees of freedom, namely a few-levels atom or a spin. In the latter case, $\mathcal{H}_x = \mathbb{C}^{2S_x+1}$ is the state space of a spin-S, with $S \in 1/2\mathbb{N}$, and it carries the $(2S_x+1)$-dimensional irreducible representation of the quantum mechanics rotation group $SU(2)$. A UHF is therefore the algebra of observables of atoms in an optical lattice or of magnetic moments of nuclei in a crystal.

A state ω on $\mathcal{A}$ has the property that it is generated by a a family of density matrices defined by: if $x\in\mathcal{A}_\Lambda$, then $\omega(x)=\mathrm{Tr}_{\mathcal{H}_\Lambda}(\rho^\omega_\Lambda x)$. Such a state is called locally normal. We have:

Proposition 15. *If ω is a state of a quantum spin system $\mathcal{A}$, then the density matrices ρ^ω_Λ obey*

1. $\rho^\omega_\Lambda\in\mathcal{A}_\Lambda$, $\rho^\omega_\Lambda\geq 0$ *and* $\mathrm{Tr}_{\mathcal{H}_\Lambda}(\rho^\omega_\Lambda)=1$
2. *the consistency condition* $\Lambda\subset\Lambda'$ *and* $x\in\mathcal{A}_\Lambda$, *then* $\mathrm{Tr}_{\mathcal{H}_\Lambda}(\rho^\omega_\Lambda x)=\mathrm{Tr}_{\mathcal{H}_{\Lambda'}}(\rho^\omega_{\Lambda'}x)$

Conversely, given a family $\{\rho_\Lambda\}_{\Lambda\in\mathcal{F}(\Gamma)}$ satisfying $(1,2)$, there is a unique state ω^ρ on $\mathcal{A}$.

Proof. Since $\mathcal{A}_\Lambda$ is a finite dimensional matrix algebra, the restriction of ω to $\mathcal{A}_\Lambda$ is given by a density matrix satisfying (1). (2) follows from the identification $\mathcal{A}_\Lambda\simeq\mathcal{A}_\Lambda\otimes 1_{\Lambda'\setminus\Lambda}$. Conversely, a family of ρ_Λ defines is a bounded linear functional on the dense subalgebra $\mathcal{A}_{\mathrm{loc}}$. Hence it extends uniquely to a linear functional on $\mathcal{A}$ with the same bound. □

Theorem 16. *Let ω_1,ω_2 be two pure states of a quantum spin system. Then ω_1 and ω_2 are equivalent if and only if for all $\epsilon>0$, there is $\Lambda\Subset\Gamma$ such that*

$$|\omega_1(x)-\omega_2(x)|\leq\epsilon\|x\|,$$

for all $x\in\mathcal{A}_{\Lambda'}$ with $\Lambda\cap\Lambda'=\emptyset$.

In other words, two pure states of a quantum spin system are equivalent if and only if they are 'equal at infinity', namely thermodynamically equal. More generally, the theorem holds – with quasi-equivalence – for any two factor states. Note that if a state is pure, then it is irreducible, i.e. $\pi(A)'=\mathbb{C}\cdot 1$, so that $\pi(A)''=\mathcal{L}(\mathcal{H})$ and $\pi(A)'\cap\pi(A)''=\mathbb{C}\cdot 1$, hence ω is a factor state.

In practice, one is given a family of vectors Ψ^i_Λ, $i=I_\Lambda$ an index set, typically the set of thermal/ground states of a finite volume Hamiltonian H_Λ on $\mathcal{H}_\Lambda$. All states $\omega^i_\Lambda:=\langle\Psi^i_\Lambda,\cdot\,\Psi^i_\Lambda\rangle$ on $\mathcal{A}_\Lambda$ can be extended to a state on $\mathcal{A}$ (by Hahn-Banach), that we still denote ω^i_Λ. The set $\mathcal{S}:=\{\omega^i_\Lambda:\Lambda\Subset\Gamma,i\in I_\Lambda\}$ is a subset of $\mathcal{E}(\mathcal{A})$, which is weakly-* compact, hence there are weak-* accumulation points, denoted ω^i_Γ, $i\in I_\Gamma$. These are usually taken as the thermodynamic thermal/ground states of the quantum spin system.

Finally, let $\Gamma=\mathbb{Z}^d$. There is a natural notion of translations on $\mathcal{A}$ which defines a group of automorphisms $\mathbb{Z}^d\ni z\mapsto\tau_z$: If $\Lambda\Subset\Gamma$ and $x\in\mathcal{A}_\Lambda$, $\tau_z(x)$ is the same observable on $\Lambda+z$. This defines an automorphism on the dense subalgebra $\mathcal{A}_{\mathrm{loc}}$, which can be extended by continuity to τ_z on all of $\mathcal{A}$. If a state is translation invariant, $\omega\circ\tau_z=\omega$ for all $z\in\mathbb{Z}$, then τ_z is unitarily implementable in the GNS representation, namely there is $\mathbb{Z}^d\ni z\mapsto U(z)$, where $U(z)$ are unitary operators on $\mathcal{H}$ with $U(z)\Omega=\Omega$, such that $\pi(\tau_z(x))=U(z)^*\pi(x)U(z)$ for all $z\in\mathbb{Z}^d$ and $x\in\mathcal{A}$. Furthermore, $\tau_{z_1+z_2}=\tau_{z_1}\circ\tau_{z_1}$ implies $U(z_1+z_2)=U(z_1)U(z_2)$.

The following proposition is usually referred to as the asymptotic abelianness of $\mathcal{A}$

Proposition 17. *Let $\mathcal{A}$ and $z\mapsto\tau_z$ be as above. Then for each $x,y\in\mathcal{A}$,*

$$\lim_{|z|\to\infty}[\tau_z(x),y]=0.$$

Proof. Exercise. □

2.3.2 Fermions: the CAR algebra

The algebra of canonical anticommutation relations (CAR) is the algebra of creation and annihilation operators of fermions

Definition 8. *Let $\mathcal{D}$ be a prehilbert space. The* CAR algebra $\mathcal{A}_+(\mathcal{D})$ *is the C^*-algebra generated by* 1 *and elements $a(f)$, $f \in \mathcal{D}$ satisfying*

$$\begin{aligned} &f \longmapsto a(f) \text{ is antilinear} \\ &\{a(f), a(g)\} = 0, \quad \{a(f)^*, a(g)^*\} = 0 \\ &\{a(f)^*, a(g)\} = \langle g, f\rangle 1 \end{aligned}$$

for all $f, g \in \mathcal{D}$.

It follows from the CAR relations that $(a(f)^*a(f))^2 = a(f)^*\{a(f), a(f)^*\}a(f) = \|f\|^2 a(f)^*a(f)$, and the C*-property then implies $\|a(f)\| = \|f\|$ so that $f \mapsto a(f)$ is a continuous map.

Proposition 18. *Let $\mathcal{D}$ be a prehilbert space with closure $\overline{\mathcal{D}} = \mathcal{H}$. Then*

1. $\mathcal{A}_+(\mathcal{D}) = \mathcal{A}_+(\mathcal{H})$
2. *$\mathcal{A}_+(\mathcal{D})$ is unique: If $\mathcal{A}_1, \mathcal{A}_2$ both satisfy the above definition, then there exists a unique *-isomorphism $\gamma : \mathcal{A}_1 \to \mathcal{A}_2$ such that $a_2(f) = \gamma(a_1(f))$ for all $f \in \mathcal{D}$*
3. *If L is a bounded linear operator in $\mathcal{H}$ and A a bounded antilinear operator in $\mathcal{H}$ satisfying*[2]

$$\begin{aligned} L^*L + A^*A = LL^* + AA^* = 1, \\ LA^* + AL^* = L^*A + A^*L = 0, \end{aligned}$$

*there is a unique *-automorphism $\gamma_{L,A}$ of $\mathcal{A}_+(\mathcal{H})$ such that $\gamma_{L,A}(a(f)) = a(Lf) + a(Af)^*$.*

Proof. Since $\mathcal{D}$ is a subset of $\mathcal{H}$, we have that $\mathcal{A}_+(\mathcal{D}) \subset \mathcal{A}_+(\mathcal{H})$. Moreover, if $f \in \mathcal{H}$, there is a sequence $f_n \in \mathcal{D}$ such that $f_n \to f$. By linearity and continuity, $\|a(f) - a(f_n)\| = \|a(f - f_n)\| = \|f - f_n\| \to 0$, showing that $a(f) \in \mathcal{A}_+(\mathcal{D})$, and $\mathcal{A}_+(\mathcal{H}) \subset \mathcal{A}(\mathcal{D})$, proving (1).

Assume now that $\dim\mathcal{H} < \infty$, and that $\{f_i\}_{i=1}^n$ is an orthonormal basis. Then the map $\mathcal{I} : \mathcal{A}_+(\mathcal{H}) \to \mathcal{M}_2^{\otimes n}$ defined by

$$\begin{aligned} \mathcal{I}(a(f_k)a(f_k)^*) = e_{11}^k \quad \mathcal{I}(V_{k-1}a(f_k)) = e_{12}^k \\ \mathcal{I}(V_{k-1}a(f_k)^*) = e_{21}^k \quad \mathcal{I}(a(f_k)^*a(f_k)) = e_{22}^k \end{aligned}$$

where e_{ij}^k is the canonical basis matrix in $\mathcal{M}_2^{\otimes n}$ which is non-trivial on the k-th factor, and $V_k = \prod_{i=1}^k (1 - 2a(f_i)^*a(f_i))$, is an algebra isomorphism. In particular, the CAR imply that $e_{ij}^k e_{ab}^k = \delta_{ja} e_{ib}^k$ and $[e_{ij}^k, e_{ab}^l] = 0$ if $k \neq l$ as it should. Furthermore, it is invertible with inverse

$$a(f_k) = \mathcal{I}^{-1}\left(\prod_{i=1}^{k-1}(e_{11}^i - e_{22}^i)e_{12}^k\right).$$

This proves (2) for the finite dimensional case. If $\mathcal{H}$ is infinite dimensional, there is a basis $\{f_\alpha\}_{\alpha\in A}$ of $\mathcal{H}$, not necessarily countable, and the above construction can be made with any finite subset of A. We conclude in this case by (1) since the vector space of finite linear combinations of f_α is dense in $\mathcal{H}$.

[2] By definition, $\langle f, Ag\rangle = \langle g, A^*f\rangle$ for an antilinear operator

Finally,

$$\{a(Lf) + a(Af)^*, a(Lg)^* + a(Ag)\} = \langle Lf, Lg\rangle + \langle Ag, Af\rangle = \langle f, g\rangle,$$

and similar computations for other anticommutators show that 1 and $a(Lf) + a(Af)^*$ for all $f \in \mathcal{H}$ also generate $\mathcal{A}_+(\mathcal{H})$, conluding the proof by (2). □

Note that the proof of (2) shows that the CAR algebra is a UHF algebra.

The transformation $\gamma_{L,A}$ is called a Bogoliubov transformation. Its unitary implementability in a given representation is a separate question, which can be completely answered in the case of so-called quasi-free representations. A particularly simple case is given by $A = 0$ and a unitary L, corresponding to the non-interacting evolution of single particles under L.

2.3.3 Bosons: the CCR algebra

The algebra of canonical commutation relations (CCR) is the algebra of creation and annihilation operators of bosons. Being unbounded operators, they do not form a C*-algebra, but their exponentials do so and it is usually referred to, in this form, as the Weyl algebra.

Definition 9. *Let $\mathcal{D}$ be a prehilbert space. The* Weyl algebra *$\mathcal{A}_-(\mathcal{D})$ is the C*-algebra generated by $W(f)$, $f \in \mathcal{D}$ satisfying*

$$\begin{aligned} W(-f) &= W(f)^* \\ W(f)W(g) &= \exp\left(-\frac{\mathrm{i}}{2}\mathrm{Im}\langle f, g\rangle\right) W(f+g) \end{aligned}$$

for all $f, g \in \mathcal{D}$.

Note the commutation relation $W(f)W(g) = \exp(-\mathrm{i}\mathrm{Im}\langle f, g\rangle)W(g)W(f)$.

Proposition 19. *Let $\mathcal{D}$ be a prehilbert space with closure $\overline{\mathcal{D}} = \mathcal{H}$. Then*

1. *$\mathcal{A}_-(\mathcal{D}) = \mathcal{A}_-(\mathcal{H})$ if and only if $\mathcal{D} = \mathcal{H}$*
2. *$\mathcal{A}_-(\mathcal{D})$ is unique: If $\mathcal{A}_1, \mathcal{A}_2$ both satisfy the above definition, then there exists a unique *-isomorphism $\gamma : \mathcal{A}_1 \to \mathcal{A}_2$ such that $W_2(f) = \gamma(W_1(f))$ for all $f \in \mathcal{D}$*
3. *$W(0) = 1$, $W(f)$ is a unitary element and $\|W(f) - 1\| = 2$ for all $f \in \mathcal{D}$, $f \neq 0$*
4. *If S is a real linear, invertible operator in $\mathcal{D}$ such that $\mathrm{Im}\langle Sf, Sg\rangle = \mathrm{Im}\langle f, g\rangle$, then there is a unique *-automorphism γ_S of $\mathcal{A}_-(\mathcal{D})$ such that $\gamma_S(W(f)) = W(Sf)$.*

In fact, $\mathcal{D}$ only needs to be a real linear vector space equipped with a symplectic form, and S is a symplectic map. This is the natural structure of phase space and its Hamiltonian dynamics in classical mechanics, and the map $f \mapsto W(f)$ is called the Weyl quantisation[3]. (3) shows in particular that it is a discontinuous map. We only prove (3) and (4). The difference between the CAR and CCR algebra with respect to closure of the underlying space is due to the lack of continuity of $f \mapsto W(f)$.

Proof. The definition implies that $W(f)W(0) = W(f) = W(0)W(f)$ so that $W(0) = 1$. Moreover, $W(f)W(-f) = W(-f)W(f) = W(0) = 1$ so that $W(f)$ is unitary. In turn, this implies

$$W(g)W(f)W(g)^* = \exp(\mathrm{i}\mathrm{Im}\langle f, g\rangle)W(f).$$

[3]In fact, it is also an algebra isomorphism between $\mathcal{D} = C^\infty(X)$ equipped with a Poisson bracket and $\mathcal{A}_-(\mathcal{D})$

Hence, the spectrum of $W(f)$ is invariant under arbitrary rotations for any $f \neq 0$, so that $\mathrm{Sp}(W(f)) = S^1$. Hence, $\sup\{|\lambda| : \lambda \in \mathrm{Sp}(W(f) - 1)\} = 2$, which concludes the proof of (3) since $W(f) - 1$ is a normal operator. Finally, (4) follows again from (2) and the invariance of the Weyl relations. □

Definition 10. *A representation $(\mathcal{H}, \pi)$ of $\mathcal{A}_-(\mathcal{D})$ is* regular *if $t \mapsto \pi(W(tf))$ is a strongly continuous map on $\mathcal{H}$ for all $f \in \mathcal{D}$.*

In a regular representation, $\mathbb{R} \ni t \mapsto \pi(W(tf))$ is a strongly continuous group of unitaries by the Weyl relations, so that Stone's theorem yields the existence of a densely defined, self-adjoint generator $\Phi_\pi(f)$ such that $\pi(W(tf)) = \exp(\mathrm{i}t\Phi_\pi(f))$ for all $f \in \mathcal{D}$. In fact, for any finite dimensional subspace $\mathcal{K} \subset \mathcal{D}$ there is a common dense space of analytic vectors of $\{\Phi_\pi(f), \Phi_\pi(\mathrm{i}f), f \in \mathcal{K}\}$, namely for which $\sum_{n=0}^\infty \|\Phi_\pi^n \psi\| t^n / n! < \infty$ for t small enough. The creation and annihilation operators can be defined

$$a_\pi^*(f) := 2^{-1/2} \left(\Phi_\pi(f) - \mathrm{i}\Phi_\pi(\mathrm{i}f)\right), \qquad a_\pi(f) := 2^{-1/2} \left(\Phi_\pi(f) + \mathrm{i}\Phi_\pi(\mathrm{i}f)\right)$$

on $D(a_\pi^*(f)) = D(a_\pi(f)) = D(\Phi_\pi(f)) \cap D(\Phi_\pi(\mathrm{i}f))$, which is dense. Note that $a_\pi^*(f) \subset a_\pi(f)^*$. In fact, equality holds.

By construction $f \mapsto \Phi_\pi(f)$ is real linear, so that $f \mapsto a_\pi(f)$ is antilinear and $f \mapsto a_\pi^*(f)$ is linear. Now, taking the second derivative of the Weyl relations applied on any vector $\xi \in D(\Phi_\pi(f)) \cap D(\Phi_\pi(g))$ at $t = t' = 0$, one obtains $\left(\Phi_\pi(f)\Phi_\pi(g) - \Phi_\pi(g)\Phi_\pi(f)\right)\xi = \mathrm{i}\mathrm{Im}\langle f, g\rangle \xi$, so that

$$\left(a_\pi(f)a_\pi^*(g) - a_\pi^*(g)a_\pi(f)\right)\xi = \langle f, g\rangle \xi$$

the usual form of the canonical commutation relations (CCR). Finally, we prove that the creation/annihilation operators are closed. Indeed, $\|\Phi_\pi(f)\xi\|^2 + \|\Phi_\pi(\mathrm{i}f)\xi\|^2 = \|a_\pi(f)\xi\|^2 + \|a_\pi^*(f)\xi\|^2$, while the commutation relations yield $\|a^*(f)\xi\|^2 - \|a(f)\xi\|^2 = \|f\|^2\|\xi\|^2$. Together, $\|\Phi_\pi(f)\xi\|^2 + \|\Phi_\pi(\mathrm{i}f)\xi\|^2 = 2\|a_\pi(f)\xi\|^2 + \|f\|^2\|\xi\|^2$. Hence, for any sequence $\psi_n \in D(a_\pi(f))$ such that $\psi_n \to \psi$ and $a_\pi(f)\psi_n$ converges, we have that $\Phi_\pi(f)\psi_n, \Phi_\pi(\mathrm{i}f)\psi_n$ converge. Since Φ_π are self-adjoint and hence closed, we have that $\psi \in D(a_\pi(f))$, and $\Phi_\pi(f)\psi_n \to \Phi_\pi(f)\psi$ and $\Phi_\pi(\mathrm{i}f)\psi_n \to \Phi_\pi(\mathrm{i}f)\psi$. By the norm equality again, $a_\pi(f)\psi_n \to a_\pi(f)\psi$ and $a_\pi(f)$ is closed.

2.3.4 Fock spaces and the Fock representation

The set $\mathcal{D}^{\otimes n}$ carries an action Π of the permutation group S_n

$$\Pi_\pi : \psi_1 \otimes \cdots \otimes \psi_n \longmapsto \psi_{\pi^{-1}(1)} \otimes \cdots \otimes \psi_{\pi^{-1}(n)}$$

for any $\pi \in S_n$ and we denote $\mathcal{D}_\pm^{(n)} := \{\Psi^{(n)} \in \mathcal{D}^{\otimes n} : \Pi_\pi \Psi^{(n)} = (\pm 1)^{\mathrm{sgn}\pi} \Psi^{(n)}\}$, namely the symmetric, respectively antisymmetric subspace of $\mathcal{D}^{\otimes n}$. Let also $\mathcal{D}_\pm^{(0)} := \mathbb{C}$. The bosonic, respectively fermionic Fock space over $\mathcal{D}$ is denoted $\mathcal{F}_\pm(\mathcal{D}) := \oplus_{n=0}^\infty \mathcal{D}_\pm^{(n)}$. That is, a vector $\Psi \in \mathcal{F}_\pm(\mathcal{D})$ can be represented as a sequence $(\Psi^{(n)})_{n \in \mathbb{N}}$ such that $\Psi^{(n)} \in \mathcal{D}_\pm^{(n)}$, with $\sum_{n \in \mathbb{N}} \|\Psi^{(n)}\| < \infty$. The vector $\Omega := (1, 0, \ldots)$ is called the vacuum. We further denote $\mathcal{F}_\pm^{\mathrm{fin}}(\mathcal{D}) := \{\Psi \in \mathcal{F}_\pm(\mathcal{D}) : \exists N \in \mathbb{N} \text{ with } \Psi^{(n)} = 0, \forall n \geq N\}$, which is dense. Note that the probability to find more than N particles in any vector Ψ vanishes as $N \to \infty$,

$$P_{\geq N}(\Psi) := \sum_{n \geq N} \|\Psi^{(n)}\|^2 \longrightarrow 0, \qquad (N \to \infty),$$

which we interpret as follows: In Fock space, there is an arbitrarily large but finite number of particles. In particular, there is no vector representing a gas at non-zero density in the thermodynamic limit. We define $N : \mathcal{F}_\pm^{\mathrm{fin}}(\mathcal{D}) \to \mathcal{F}_\pm^{\mathrm{fin}}(\mathcal{D})$ by $N\Psi = n\Psi$ whenever $\Psi \in \mathcal{D}_\pm^{(n)}$.

For $f \in \mathcal{D}$, let $b_\pm(f) : \mathcal{D}^{\otimes n} \to \mathcal{D}^{\otimes n-1}$ be defined by

$$b_\pm(f)(\psi_1, \dots \psi_n) = \sqrt{n}\langle f, \psi_1\rangle(\psi_2, \dots \psi_n),$$

which maps $\mathcal{D}_\pm^{(n)}$ to $\mathcal{D}_\pm^{(n-1)}$, with $b_\pm(f)\mathcal{D}_\pm^{(0)} = 0$, and hence $b_\pm(f) : \mathcal{F}_\pm(\mathcal{D}) \to \mathcal{F}_\pm(\mathcal{D})$. Its adjoint $b_\pm^*(f) := b_\pm(f)^* : \mathcal{D}_\pm^{(n-1)} \to \mathcal{D}_\pm^{(n)}$ such that

$$b_\pm^*(f)\Psi^{(n-1)} = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\pm 1)^{k-1}\Pi_{\pi_k} f \otimes \Psi^{(n-1)}$$

where $\pi_k^{-1} = (k, 1, 2, \dots, k-1, k+1, \dots, n)$. Indeed, the right hand side $\tilde\Psi$ is in $\mathcal{D}_\pm^{(n)}$: $(\pi_{\sigma(k)} \circ \sigma^{-1} \circ \pi_k^{-1})(1) = 1$ and the signature of the permutation is $(k-1) + \mathrm{sgn}(\sigma) + (\sigma(k) - 1)$, so that $\Pi_{\pi_{\sigma(k)}^{-1}}\Pi_\sigma\Pi_{\pi_k}(f \otimes \Psi^{(n-1)}) = (\pm 1)^{\mathrm{sgn}(\sigma)+(\sigma(k)-k)} f \otimes \Psi^{(n-1)}$, which implies that $\Pi_\sigma\tilde\Psi = (\pm 1)^{\mathrm{sgn}(\sigma)}\tilde\Psi$. Moreover, for any $\Upsilon^{(n)} \in \mathcal{D}_\pm^{(n)}$,

$$\langle b_\pm(f)\Upsilon^{(n)}, \Psi^{(n-1)}\rangle = \sqrt{n}\langle \Upsilon^{(n)}, f \otimes \Psi^{(n-1)}\rangle = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\langle \Pi_{\pi_k}\Upsilon^{(n)}, \Pi_{\pi_k}(f \otimes \Psi^{(n-1)})\rangle = \langle \Upsilon^{(n)}, \tilde\Psi\rangle$$

where we used that $\Pi_{(\cdot)}$ is unitary, proving that $\tilde\Psi = b_\pm^*(f)\Psi^{(n-1)}$.

Proposition 20. *1. $f \mapsto b_\pm(f)$ is antilinear, $f \mapsto b_\pm^*(f)$ is linear*

2. $Nb_\pm(f) = b_\pm(f)(N-1)$

3. $b_\pm(f), b_\pm^*(g)$ satisfy the canonical commutation, resp. anticommutation relations

Proof. We denote $[A, B]_\pm := AB \mp BA$ and prove $[b_\pm(f), b_\pm^*(g)]_\pm = \langle f, g\rangle$. Indeed, for $\Psi^{(n-1)} = \psi_1 \otimes \cdots \otimes \psi_{n-1}$,

$$\frac{1}{\sqrt{n}}b_\pm(f)\Pi_{\pi_{k+1}}(g\otimes\Psi^{(n-1)}) = \langle f, \psi_1\rangle\psi_2\otimes\cdots\otimes\psi_k\otimes g\otimes\cdots\psi_{n-1} = \frac{1}{\sqrt{n-1}}\Pi_{\pi_k}(g\otimes b_\pm(f)\Psi^{(n-1)}),$$

so that

$$\begin{aligned} b_\pm(f)b_\pm(g)\Psi^{(n-1)} &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\pm 1)^{k-1}b_\pm(f)\Pi_{\pi_k}(g \otimes \Psi^{(n-1)}) \\ &= \langle f, g\rangle\Psi^{(n-1)} \pm \frac{1}{\sqrt{n-1}}\sum_{k=1}^{n-1}(\pm 1)^{k-1}\Pi_{\pi_k}(g \otimes b_\pm(f)\Psi^{(n-1)}) \\ &= \langle f, g\rangle\Psi^{(n-1)} \pm b_\pm(g)b_\pm(f)\Psi^{(n-1)}, \end{aligned}$$

where the second equality follows by extracting the first term in the sum and using the observation above in the remaining terms. □

In particular, $\{b_-(f) : f \in \mathcal{D}\}$ form a representation of the CAR algebra. Furthermore, The operators $\Phi_+(f) := 2^{-1/2}(b_+(f) + b_+^*(f))$ are symmetric on $\mathcal{F}_+^{\mathrm{fin}}(\mathcal{D})$ and extend to self-adjoint operators, so that $W_+(f) := \exp(\mathrm{i}\Phi_+(f))$ are well-defined unitary operators on $\mathcal{F}_+(\mathcal{D})$, yielding a representation of the Weyl algebra. They are the fermionic and bosonic Fock representations associated to the Fock state

$$\begin{cases} \omega_F^{\mathrm{CAR}}(a(f)^*a(g)) := \left\langle \Omega, b_-^*(f)b_-(g)\Omega \right\rangle = 0 \text{ and } \omega_F^{\mathrm{CAR}}(a(f)) := 0 & \text{(fermions)} \\ \omega_F^{\mathrm{CCR}}(W(f)) := \langle \Omega, W_+(f)\Omega\rangle = \mathrm{e}^{-\|f\|^2/4} & \text{(bosons)} \end{cases}$$

In other words, Fock spaces are the GNS Hilbert spaces for the Fock states.

Quantum mechanics in one dimension for one particle is usually associated with the Schrödinger representation, defined on the Hilbert space $L^2(\mathbb{R})$. It arises as the regular representation of the Weyl algebra $\mathcal{A}_-(\mathbb{C})$ given by

$$\pi_S(W(s+\mathrm{i}t)) := \mathrm{e}^{\frac{\mathrm{i}}{2}st}U(s)V(t),$$

where

$$(U(s)\psi)(x) = \mathrm{e}^{\mathrm{i}st}\psi(x), \qquad (V(t)\psi)(x) = \psi(x+t),$$

with self-adjoint generators $X := \Phi_S(1)$ and $P := \Phi_S(\mathrm{i}) = -\mathrm{i}\partial_x$.

In fact, $L^2(\mathbb{R})$ carries a Fock space structure, obtained by introducing $a_S := 2^{-1/2}(X+\mathrm{i}P)$ and $a_S^* := 2^{-1/2}(X-\mathrm{i}P)$, which satisfy the CCR (strongly on a dense set such as $C_c^\infty(\mathbb{R})$. The vacuum vector Ω_S is the L^2-normalised solution of $a_S\Omega_S = 0$, namely

$$(x+\partial_x)\Omega_S(x) = 0, \qquad \text{i.e.} \qquad \Omega_S(x) = \pi^{-1/4}\mathrm{e}^{-x^2/2}.$$

With $\mathcal{H}^n := \mathrm{span}\{(a_S^*)^n\Omega_S\}$, namely the span of the nth Hermite function, one obtains $L^2(\mathbb{R}) = \oplus_{n=0}^\infty\mathcal{H}^n$. In other words, $L^2(\mathbb{R}) \simeq \mathcal{F}_+(\mathbb{C})$ and the Schrödinger and Fock representations are equivalent, the unitary map being $(a_S^*)^n\Omega_S \mapsto (b_+^*)^n\Omega$.

Hence, the dimension of $\mathcal{D}$ has the interpretation of 'the number of degrees of freedom' of the system and N-body quantum mechanics in $\mathbb{R}^d$ corresponds to the algebra $\mathcal{A}_-(\mathbb{C}^{Nd})$, which has a Schrödinger representation, namely the (Nd)-fold tensor product representation of that given above. In fact, this is the only one:

Theorem 21. *Let $\mathcal{H}$ be a finite dimensional Hilbert space,* $\dim\mathcal{H} = n$*. Then, any irreducible representation of $\mathcal{A}_-(\mathcal{H})$ is equivalent to the Schrödinger representation.*

In other words, the algebraic machinery is useless in quantum mechanics. Whenever $\dim\mathcal{H} = \infty$, typically $\mathcal{H} = L^2(\mathbb{R}^d)$ itself, there are truly inequivalent representations: these are in particular those arising in quantum statistical mechanics.