Problem Sheet 8: Solution

(No responsibility is taken for the correctness of the solutions.)

Ex. 1: We take for granted that $-\Delta$ is self-adjoint on $H^{2,2}(\mathbb{R}^3)$ and essentially self-adjoint on $C_c^\infty(\mathbb{R}^3)$. The claim will follow from the Kato-Rellich theorem once we prove that V is relatively $-\Delta$ bounded with relative bound < 1. We will in fact prove that the relative bound is 0. Without loss of generality we may assume $V_2 = 0$ and we write V instead of V_1. By Sobolev embedding,

$$\|\psi\|_{L^\infty} \le C_s \|\psi\|_{H^{s,2}}, \psi \in H^{s,2}(\mathbb{R}^3),$$

for any $s > 3/2$. We fix $s \in (3/2, 2)$ and estimate[1] for $\xi \in \mathbb{R}^3$

$$(1+|\xi|^2)^s \le (1+\lambda^2)^s + \frac{(1+\lambda^2)^s}{\lambda^4}|\xi|^4,$$

where $\lambda > 1$ is arbitrary and will be fixed momentarily. We then have for $\psi \in H^{2,2}(\mathbb{R}^3)$,

$$\|\psi\|_{H^{s,2}}^2 = \int (1+|\xi|^2)^s |\widehat{\psi}(\xi)|^2 \, \mathrm{d}\xi \le (1+\lambda^2)^s \|\psi\|_{L^2}^2 + \frac{(1+\lambda^2)^s}{\lambda^4}\|\Delta\psi\|_{L^2}^2$$

Given $\epsilon > 0$, choose λ such that

$$C_s^2 \|V\|_{L^2}^2 \frac{(1+\lambda^2)^s}{\lambda^4} < \epsilon^2.$$

Note that this is possible since $s < 2$. Then

$$\|V\psi\|_{L^2}^2 \le \|V\|_{L^2}^2 \|\psi\|_{L^\infty}^2 \le C_s^2 \|V\|_{L^2}^2 \|\psi\|_{H^{s,2}}^2 \le C_s^2 \|V\|_{L^2}^2 (1+\lambda^2)^s \|\psi\|_{L^2}^2 + \epsilon^2 \|\Delta\psi\|_{L^2}^2.$$

Since ϵ is arbitrary, the claim is proved.

Ex. 2: (i) We have for every $\lambda \in \mathbb{C}$

$$0 \le \omega((A+\lambda B)^*(A+\lambda B)) = \omega(A^*A) + \overline{\lambda}\omega(B^*A) + \lambda\omega(A^*B) + |\lambda|^2\omega(B^*B). \tag{1}$$

For $\lambda = 1$ and for $\lambda = \mathrm{i}$, this becomes

$$\begin{aligned} 0 &\le \omega(A^*A) + \omega(B^*A) + \omega(A^*B) + \omega(B^*B), \\ 0 &\le \omega(A^*A) - \mathrm{i}\omega(B^*A) + \mathrm{i}\omega(A^*B) + \omega(B^*B), \end{aligned}$$

[1] Write $(1+\lambda^2)^s = \mathbf{1}\{|\xi| \le \lambda\}(1+\lambda^2)^s + \mathbf{1}\{|\xi| > \lambda\}(1+\lambda^2)^s$.

respectively. This implies that

$$\begin{aligned}\mathrm{Im}\omega(B^*A) + \mathrm{Im}\omega(A^*B) &= 0,\\ \mathrm{Re}\omega(B^*A) - \mathrm{Re}\omega(A^*B) &= 0,\end{aligned}$$

and this is equivalent to $\overline{\omega(B^*A)} = \omega(A^*B)$. Setting $B = 1$ yields

$$\overline{\omega(A)} = \omega(A^*). \tag{2}$$

Next, let $\lambda \in \mathbb{R}$. Then (1) and the result just proved imply that the discriminant of the polynomial on the right, i.e. of

$$\lambda \mapsto \omega(A^*A) + 2\lambda\mathrm{Re}\,(\omega(A^*B)) + \omega(B^*B)$$

must be nonpositive; equivalently,

$$\mathrm{Re}\,(\omega(A^*B))^2 \leq \omega(A^*A)\omega(B^*B).$$

Similarly, using (1) with λ replaced by $c\lambda$, where $|c| = 1$ and $\lambda \in \mathbb{R}$, we get

$$\mathrm{Re}\,(c\omega(A^*B))^2 \leq \omega(A^*A)\omega(B^*B).$$

We may choose c such that $|\omega(A^*B)| = c\omega(A^*B)$, in which case $\mathrm{Re}\,(c\omega(A^*B))^2 = |\omega(A^*B)|^2$. This proves that

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B). \tag{3}$$

Since[2]

$$B^*B - \|B\|^2 1 \leq 0 \implies A^*B^*BA - \|B\|^2 A^*A \leq 0,$$

it follows from the positivity of ω that

$$\omega((BA)^*BA) \leq \|B\|^2\omega(A^*A). \tag{4}$$

Combining (2)–(4) gives

$$|\omega(A^*BA)|^2 \leq \omega(A^*BB^*A)\omega(A^*A) \leq \omega(A^*A)^2\|B\|^2.$$

(ii) Let $A \in \mathcal{A}$, $N \in \mathcal{N}$. Then by (4),

$$\omega((AN)^*AN) \leq \omega(N^*N)\|A\|^2 = 0.$$

[2]The first inequality follows from $\sigma(B^*B - \|B\|^2 1) = \sigma(B^*B) - \|B\|^2 \subseteq [-r(B^*B), r(B^*B)] - \|B\|^2 = [-2\|B\|^2, 0]$. The second follows from the fact that $C \geq 0 \iff \exists D$ s.t. $C = D^*D$ (take $-C = B^*B - \|B\|^2 1$).

(iii) By (2) $\mathcal{N}$ is closed under taking adjoints. Then (abusing notation)

$$\omega((A+\mathcal{N})^*(B+\mathcal{N})) = \omega(A^*B) + \omega(A^*\mathcal{N}) + \omega(\mathcal{N}^*B) + \omega(\mathcal{N}^*\mathcal{N}) = \omega(A^*B).$$

Hence the bilinear form is well-defined. Positivity immediately follows from positivity of ω. Symmetry follows from (2).

(iv) By the same argument following (3) we have

$$\omega(A^*A) \leq \|A\|^2. \tag{5}$$

Let $A \in \mathcal{A}$, $\psi_B \in h$. By (4) and (5),

$$\|\pi(A)\psi_B\|_{\mathcal{H}}^2 = \langle \psi_{AB}, \psi_{AB} \rangle = \omega((AB)^*AB) \leq \|A\|^2 \omega(B^*B).$$

Since h is dense in $\mathcal{H}$ (by definition) this implies that

$$\|\pi(A)\|_{\mathcal{L}(\mathcal{H})} \leq \|A\|,$$

i.e. π is bounded. Moreover,

$$\langle \psi_B, \pi(A^*)\psi_C \rangle = \langle \psi_B, \psi_{A^*C} \rangle = \omega(B^*A^*C) = \langle \psi_{AB}, \psi_C \rangle = \langle \pi(A)\psi_B, \psi_C \rangle$$

shows that $\psi(A^*) = \pi(A)^*$, and

$$\pi(AB)\psi_C = \psi_{ABC} = \pi(A)\pi(B)\psi_C$$

shows that $\pi(AB) = \pi(A)\pi(B)$.

We have

$$\omega(A) = \langle \psi_1, \psi_A \rangle = \langle \Omega, \pi(A)\Omega \rangle.$$

Ex. 3: It suffices to compute

$$\mathcal{N} = \{A \in \mathcal{A} : \omega(A^*A) = 0\} = \begin{cases} \left\{ \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \right\} & \text{if } \alpha = 0, \\ \{0\} & \text{if } \alpha \in (0,1), \\ \left\{ \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \right\} & \text{if } \alpha = 1. \end{cases}$$

One easily checks that the GNS representation is irreducible iff $\alpha \in \{0,1\}$ (pure states). Any state is a vector state in its own GNS representation.