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Mathematical Quantum Mechanics

Problem Sheet 9

Hand-in deadline: 20.12.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: For $\mathcal{H}_a$ and $\mathcal{H}_b$ finte dimensional Hilbert spaces, the tensor product $\mathcal{H}_a \otimes \mathcal{H}_b$ is the Hilbert space spanned by elements $\psi_a \otimes \psi_b$ for $\psi_a \in \mathcal{H}_a$ and $\psi_b \in \mathcal{H}_b$ up to relations

$$\begin{aligned}(\lambda\psi_a) \otimes \psi_b = \lambda(\psi_a \otimes \psi_b) &= \psi_a \otimes (\lambda\psi_b)\\ \psi_a \otimes \psi_b + \tilde{\psi}_a \otimes \psi_b &= (\psi_a + \tilde{\psi}_a) \otimes \psi_b\\ \psi_a \otimes \psi_b + \psi_a \otimes \tilde{\psi}_b &= \psi_a \otimes (\psi_b + \tilde{\psi}_b)\end{aligned}$$

with the scalar product defined by

$$\langle \psi_a \otimes \psi_b, \tilde{\psi}_a \otimes \tilde{\psi}_b \rangle = \langle \psi_a, \tilde{\psi}_a \rangle \langle \psi_b, \tilde{\psi}_b \rangle$$

and extended by linearity. This implies that if $e_1, \ldots, e_n$ is an ON basis of $\mathcal{H}_a$ and $f_1, \ldots, f_m$ is an ON basis of $\mathcal{H}_b$, and ON basis of $\mathcal{H}_a \otimes \mathcal{H}_b$ is given by $(e_i \otimes f_j)_{ij}$, in particular the dimension of the tensor product is the product of dimensions of the factors.

$\mathcal{B}(\mathcal{H}_a \otimes \mathcal{H}_b)$ is spanned by elements $A \otimes B$ for $A \in \mathcal{B}(\mathcal{H}_a)$ and $B \in \mathcal{B}(\mathcal{H}_b)$ with

$$(A \otimes B)(\psi_a \otimes \psi_b) = (A\psi_a) \otimes (B\psi_b).$$

Physically, this describes the situation that one has two subsystems a (associated to Alice) and b (associated to Bob).

(i) An observable in Alice's subsystem is also an observable of the joint system, $\mathcal{B}(\mathcal{H}_a) \hookrightarrow \mathcal{B}(\mathcal{H}_a \otimes \mathcal{H}_b)$ via $A \mapsto A \otimes 1$. Thus any state $\omega\colon \mathcal{B}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathbb{C}$ (given in terms of a density matrix) descends by restriction to a state ω_a of $\mathcal{B}(\mathcal{H}_a)$:

$$\omega_a(A) := \omega(A \otimes 1).$$

Show that the density matrix of this reduced state is given by the partial trace

$$\varrho_a = \mathrm{tr}_{\mathcal{H}_b} \varrho = \sum_{j=1}^{m} \langle f_j, \varrho f_j \rangle.$$

(ii) Give an example of a pure state of $\mathcal{B}(\mathcal{H}_a \otimes \mathcal{H}_b)$ whose restriction to $\mathcal{B}(\mathcal{H}_a)$ is not pure. Show that this is not possible for commutative algebras. (Pure states are those where one has maximal information about the state of the system. This exercise shows that in quantum physics maximal information about a system does not imply maximal information about its subsystems. This is known as entanglement)

(iii) Show that any (possibly mixed) state of $\mathcal{B}(\mathcal{H}_a)$ can be obtained by a reduction from a pure state of $\mathcal{B}(\mathcal{H}_a \otimes \mathcal{H}_b)$ if $\dim \mathcal{H}_a \leq \dim \mathcal{H}_b$. (This is called 'purification' and implie that all mixed states can be thought of as arising from pure states of an enlarged system).

Exercise 2: For $\Lambda \in \mathbb{N}$, let

$$\mathcal{A}_\Lambda := \bigotimes_{i=-\Lambda}^{\Lambda} Mat(2 \times 2, \mathbb{C})_i$$

Clearly, for $\Lambda < \Lambda'$, we have $\mathcal{A}_\Lambda \hookrightarrow \mathcal{A}_{\Lambda'}$ by extending by the indentity matrix for $\Lambda < |i| \leq \Lambda'$. We can take the inductive limit as a *-algebra

$$\mathcal{A}_{loc} = \{f \colon \mathbb{Z} \to Mat(n \times x, \mathbb{C}) \mid f(k) = 1 \,\text{for almost all}\, k \in \mathbb{Z}\}\,.$$

$\mathcal{A}_{loc}$ inherits the operator norm from the $\mathcal{A}_\Lambda$ and can be closed to $\mathcal{A}$, the algebra of 'quasi-local operators'. This is a C*-algebra (you don't need to prove this). It is called a spin chain in the physics literature.

The goal of this exercise is to show that $\mathcal{A}$ admits two inequivalent represenations.

Let $\omega_\Lambda^\pm \colon A_\Lambda \to \mathbb{C}$ be given by

$$\omega_\Lambda^+(a_{-\Lambda} \otimes \cdots \otimes a_\Lambda) = \prod_{k=-\Lambda}^{\Lambda} (a_k)_{11}$$

and

$$\omega_\Lambda^-(a_{-\Lambda} \otimes \cdots \otimes a_\Lambda) = \prod_{k=-\Lambda}^{\Lambda} (a_k)_{22}$$

the products of the top left (resp. bottom right) matrix elements. Show that these are states and that they respect the embedding for $\Lambda < \Lambda'$ and thus give rise to states $\omega_\infty^\pm$ on $\mathcal{A}$. Let $(\mathcal{H}_\pm, \rho_\pm, \Omega_\pm)$ be the corresponding GNS triples.

Denote by

$$M_\Lambda := \frac{1}{2\Lambda + 1} \sum_{k=-\Lambda}^{\Lambda} 1 \otimes \cdots \otimes 1 \otimes \sigma_k^z \otimes 1 \otimes \cdots \otimes 1$$

the average magnetization and show that $\rho_\pm(M_\Lambda) \to \pm 1$ weakly as $\Lambda \to \infty$ in the operator sense, i.e. for every $\phi, \psi \in \mathcal{H}_\pm$,

$$\lim_{\Lambda \to \infty} \langle \phi, \rho_\pm(M_\Lambda)\psi \rangle = \pm \langle \phi, \psi \rangle.$$

Conclude that $\rho_\pm$ are inequivalent representations (Hint: the spectrum is a unitary invariant). Argue that $\mathcal{A}$ in fact admits infinitely many inequivalent representations.

Exercise 3: Let $\mathcal{H} = \mathbb{C}^2$. Prove: All Hermitian $\rho \in \mathcal{B}(\mathcal{H})$ are of the form

$$\rho = \frac{1}{2}\left(c\mathbf{1}_{2\times 2} + \vec{a} \cdot \vec{\sigma}\right),$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, $c \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^3$. For which $c, \vec{a}$ is ρ a density matrix? For which is it a pure state?