Mathematisches Institut
LMU München
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Mathematical Quantum Mechanics

Problem Sheet 8

Hand-in deadline: 13.12.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: Let $V_1 \in L^2(\mathbb{R}^3)$, $V_2 \in L^\infty(\mathbb{R}^3)$. Prove that $-\Delta + V$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^3)$ and self-adjoint on $H^{2,2}(\mathbb{R}^3)$. In particular, verify that the assertions hold for the Hamiltonian of the hydrogen atom, $-\Delta - Z/|x|$. *Hint:* Use the Sobolev embedding $H^{2,2}(\mathbb{R}^3) \hookrightarrow C_0(\mathbb{R}^3)$ (that was discussed in the tutorial).

Exercise 3: Let $\mathcal{A}$ be a C*-algebra with a unit, denoted 1, and ω be a state, i.e. a positive linear functional over $\mathcal{A}$ such that $\omega(1) = 1$.

(i) Prove the following identities:

$$\omega(A^*) = \overline{\omega(A)}, \qquad |\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B),$$
$$|\omega(A^*BA)| \leq \omega(A^*A)\|B\|$$

(Hint: consider the quadratic form $\lambda \mapsto \omega((A + \lambda B)^*(A + \lambda B)))$

(ii) Let $\mathcal{N} := \{A \in \mathcal{A} : \omega(A^*A) = 0\}$. Prove that $A \in \mathcal{A}, N \in \mathcal{N}$ implies $AN \in \mathcal{N}$, i.e. $\mathcal{N}$ is a left ideal

(iii) Let $h := \mathcal{A}/\mathcal{N}$, and denote ψ_A the equivalence class of $A \in \mathcal{A}$, namely $\psi_A := \{\tilde{A} \in \mathcal{A} : \exists N \in \mathcal{N} : \tilde{A} = A + N\}$. Prove that the bilinear form over h

$$(\psi_A, \psi_B) \longmapsto \langle \psi_A, \psi_B \rangle := \omega(A^*B)$$

is well-defined (i.e. the r.h.s. is independent of the chosen representative of the classes ψ_A, ψ_B) and defines a scalar product

(iv) Hence h equipped with $\langle \cdot, \cdot \rangle$ is a pre-Hilbert space. Let $\mathcal{H}$ denote its completion. Prove that the linear map $\pi : \mathcal{A} \to \mathcal{L}(h)$ defined by

$$\pi(A)\psi_B := \psi_{AB}$$

is bounded, and that it is a *-homomorphism, namely

$$\pi(A^*) = \pi(A)^*, \qquad \pi(AB) = \pi(A)\pi(B).$$

Exercise 4: Let $\mathcal{A}$ be the C^*-algebra $\mathrm{Mar}_{2\times 2}(\mathbb{C})$ of complex 2×2 matrices with its usual linear and algebraic structure and with operator norm. Consider the density matrix

$$\rho_\alpha = \begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

depending on the parameter $\alpha \in [0, 1/2]$. Construct explicitly the GNS representation of $\mathcal{A}$ associated with the state $\omega_\alpha := \mathrm{tr}(\rho_\alpha \cdot)$ in case of (i) $\alpha = 0$, (ii) $\alpha = 1/4$, (iii) $\alpha = 1/2$. Determine whether ω_α is pure or mixed and whether it is a vector state for the respective representation.