Mathematisches Institut
LMU München
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Mathematical Quantum Mechanics

Problem Sheet 7

Hand-in deadline: 06.12.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: Let A be a closed operator in $\mathcal{H}$. Prove: If B is A-compact (i.e. relatively compact with respect to A), then it is A-bounded with relative bound 0.

Exercise 2: (a) Verify that $\frac{2}{|x|} = \sum_{j=1}^{3}[\partial_j, \frac{x_j}{|x|}]$ on smooth functions in $\mathbb{R}^3$, and use it to prove the inequality

$$\int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x|}\,\mathrm{d}x \le \|\nabla\varphi\|_2\|\varphi\|_2, \quad \text{for} \quad \varphi \in C_c^\infty(\mathbb{R}^3).$$

(b) Extend the inequality in (a) to all $\varphi \in H^1(\mathbb{R}^3)$.[1] You may use without proof that $C_c^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$.

(c) Prove that for all $Z \geq 0$,

$$\int_{\mathbb{R}^3} |\nabla\varphi(x)|^2\,\mathrm{d}x - Z\int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x|}\,\mathrm{d}x \ge -\frac{1}{4}Z^2\|\varphi\|_2^2, \quad \text{for} \quad \varphi \in H^1(\mathbb{R}^3),$$

and find $\varphi \in H^1(\mathbb{R}^3)$ which saturates the inequality.

(d) Prove that $|x|^{-1}$ is Δ-bounded. *Hint:* Use an argument analogous (a)–(b).

Exercise 3: Let $V : \mathbb{R}^n \to \mathbb{R}$ be a bounded potential such that $V(x) \to 0$ as $|x| \to \infty$. Prove that $\sigma_{\mathrm{ess}}(-\Delta + V) = [0, \infty)$.

Exercise 4: Let A, B be operators on $\mathcal{H}$ such that $\mathcal{D}(A) \subset \mathcal{D}(B)$. Show that the following statements are equivalent:

i) There exist $a, b \geq 0$ such that for all $\psi \in \mathcal{D}(A)$

$$\|B\psi\| \le a\|A\psi\| + b\|\psi\|. \tag{1}$$

[1] Recall that $H^1(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) : \partial_j f \in L^2(\mathbb{R}^3) \text{ for } j = 1, 2, 3\}$ where $\partial_j f$ is the weak derivative. The norm in $H^1(\mathbb{R}^3)$ is given by $\|f\|_{H^1(\mathbb{R}^3)}^2 = \|f\|_2^2 + \sum_{j=1}^3 \|\partial_j f\|_2^2$.

ii) There exist $a', b' \geq 0$ such that for all $\psi \in \mathcal{D}(A)$

$$\|B\psi\|^2 \leq a'^2 \|A\psi\|^2 + b'^2 \|\psi\|^2. \tag{2}$$

Moreover, show that

$$\inf\{b : \exists\, a \geq 0 \text{ such that (1) holds}\} = \inf\{b' : \exists\, a' \geq 0 \text{ such that (2) holds}\}.$$