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Mathematical Quantum Mechanics

Problem Sheet 6

Hand-in deadline: 29.11.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: Consider a C^* algebra $\mathcal{A}$ with unit element I, and define the spectrum of an element $a \in \mathcal{A}$ by

$$\sigma(a) := \{\lambda \in \mathbb{C} : (\lambda I - a) \text{ has no inverse in } \mathcal{A}\}.$$

The spectral radius of a is defined as

$$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

(i) Prove that[1]

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \leq \|a\|,$$

In particular, the above limit always exists and $\sigma(A)$ is non-empty.
Hint: To prove $r(A) \leq \liminf_n \|a^n\|^{1/n}$, study the series $\lambda^{-1}\sum_n (a/\lambda)^n$. To prove $r(a) \geq \limsup_n \|a^n\|^{1/n}$, determine the radius of convergence of this series.

(ii) Assume that a is normal, i.e. $aa^* = a^*a$. Prove that $r(a) = \|a\|$.

(iii) Prove that the norm which turns a $*$-algebra into a C^*-algebra, when it exists, is unique.

(iv) Let $\tau : \mathcal{A} \to \mathcal{B}$ be a $*$-algebra morphism, i.e. a linear map such that $\tau(A^*B) = \tau(A)^*\tau(B)$ for all $A, B \in \mathcal{A}$. Show that $\|\tau(A)\| \leq \|A\|$ for all $A \in \mathcal{A}$. *Hint:* Prove this first for selfadjoint elements and use the C^* property.

Exercise 2: Consider the set $\mathrm{Mat}_{n,n}$ of $n \times n$ complex complex matrices with its usual vector space structure. This becomes an associative algebra with

[1] For this exercise you may use that that the map $\rho(a) \ni \lambda \mapsto (a - \lambda)^{-1} \in \mathcal{A}$ is holomorphic and ignore all difficulties related to the fact that it takes values in a Banach space other than $\mathbb{C}$.

the standard matrix multiplication. Banach spaces $\mathcal{A} := (\mathrm{Mat}_{n,n}, \|\cdot\|_{\mathrm{op}})$, $\mathcal{B} := (\mathrm{Mat}_{n,n}, \|\cdot\|_{\mathrm{HS}})$ are obtained by choosing the norms

$$\|A\|_{\mathrm{op}} := \sup\{\|Ax\| : x \in \mathbb{C}^n, \|x\|_1\}, \quad \|x\| := \left(\sum_{j=1}^{n} x_j^2\right)^{1/2},$$
$$\|A\|_{\mathrm{HS}} := (\mathrm{tr}\, A^*A)^{1/2}.$$

Does the involution defind by $A^* =$ adjoint of A render $\mathcal{A}, \mathcal{B}$ into C^*-algebras?

Exercise 3: Let X be a noncompact, locally compact Hausdorff space and consider the set $C_b(X)$ of bounded continuous functions $f : X \to \mathbb{C}$ with pointwise addition and multiplication and equipped with the norm

$$\|f\| := \sup_{x \in X} |f(x)|.$$

Is $C_b(X)$ with this structure a C^*-algebra?

Exercise 4: Let $\mathcal{A}$ be an associative algebra and define a new product by

$$a \circ b := \frac{1}{2}(ab + ba).$$

Prove that $(\mathcal{A}, \circ)$ is associative if and only if

$$[\mathcal{A}, [\mathcal{A}, \mathcal{A}]] = \{0\}.$$