MATHEMATISCHES INSTITUT
LMU MÜNCHEN
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Mathematical Quantum Mechanics

Problem Sheet 5

Hand-in deadline: 22.11.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: (a) Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $\lambda \in \mathbb{R}$. Prove that the following assertions are equivalent:

(i) $\lambda \in \sigma_{\mathrm{ess}}(A)$,

(ii) There exists a sequence[1] $(\psi)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ such that

$$\liminf_{n\to\infty} \|\psi_n\| > 0, \quad \mathrm{w-}\lim_{\mathrm{n}\to\infty} \psi_{\mathrm{n}} = 0, \quad \lim_{\mathrm{n}\to\infty} \|(\mathrm{A}-\lambda)\psi_{\mathrm{n}}\| = 0.$$

Here w$-$ lim denotes the weak limit.

Exercise 2: Prove Theorem 2.11 (a) from the lecture, using Corollary 2.3. *Hint:* First replace $C(H+\mathrm{i})^{-1}$ by a finite rank operator. Then use an approximation argument.

Exercise 3: Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and assume that $\varphi \in \mathcal{H}$, $\psi \in \mathcal{H}_{\mathrm{ac}}(A)$. Prove that there exists a density $h \in L^1(\mathbb{R})$ such that

$$\mu_{\varphi,\psi}(B) = \int_B \mathrm{d}\lambda h(\lambda), \quad B \in \mathfrak{B}(\mathbb{R}).$$

Hint: Use polarization and the Radon-Nikodym theorem for (non-negative) measures.

Exercise 4: Consider the Laplacian on the torus, i.e. $-\Delta$ in $L^2([0,1]^d)$ with periodic boundary conditions. Prove that $(-\Delta+1)^{-1} \in \mathcal{T}^p(L^2([0,1]^d))$ if and only if $p > d/2$.

[1] also called a *Weyl sequence* or *singular sequence* (for the essential spectrum); compare Exercise 4 on Sheet 2.