Mathematisches Institut
LMU München
WS 2017-18

Prof. P. Müller
Dr. R. Helling
Dr. J.-C. Cuenin

Mathematical Quantum Mechanics

Problem Sheet 4

Hand-in deadline: 15.11.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: Let $A = A^* \in BL(\mathcal{H})$ and $0 \neq \psi \in \mathcal{H}$, and define the corresponding spectral measure $\mu_\psi := \langle \psi, \mathbf{1}_\bullet(A)\psi \rangle$. Prove that $\psi \in \mathcal{H}_{\mathrm{ac}}(A)$ if and only if $\mu_\psi = \mu_{\psi,\mathrm{ac}}$ (i.e. μ_ψ is absolutely continuous with respect to Lebesgue measure).

Exercise 2: Let $A = A^*$ and $a, b \in \mathbb{R}$ with $a < b$. Show that the following are equivalent:

(i) $(a, b) \subset \rho(A)$,

(ii) $\|2A\psi - (a + b)\psi\| \geq (b - a)\|\psi\|$ for all $\psi \in \mathcal{D}(A)$,

(iii) $(A - bI)(A - aI) \geq 0$.

Exercise 3: Let $H = H^*$, $f \in \mathcal{H}$ and $\omega \in \mathbb{R}$. Consider the initial-value problem

$$\mathrm{i}\partial_t u + Hu = -f\mathrm{e}^{\mathrm{i}\omega t}, \quad u(0) = 0 \tag{1}$$

for $u \in C^1(\mathbb{R}; \mathcal{H})$. Define the energy $E(t) := \|u(t)\|^2$.

(a) Show that (1) has a unique solution $u \in C^1(\mathbb{R}; \mathcal{H})$ given by

$$u(t) = (\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{\mathrm{i}Ht})(\omega - H)^{-1} f.$$

(b) Suppose that $\omega \in \rho(H)$. Show that $E(\cdot)$ is bounded.

(c) Suppose that $Hf = \omega f$. Show that $E(t) = t^2$.

(d) Suppose that $\omega \in \sigma_{\mathrm{ac}}(H)$ and that $f \in \mathcal{H}_{\mathrm{ac}}$ has a density that is continuous at ω. Show that there exists $c > 0$ such that $\lim_{t\to\infty} E(t)/t = c$.

Exercise 4: Let $H = -\Delta$, $\mathcal{D}(H) = H^{2,2}(\mathbb{R}^d)$.

(a) Determine the spectral measure $\mathbf{1}_\bullet(H)$.

(b) Prove that $\sigma(H) = \sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ac}}(H) = [0, \infty)$.