Mathematisches Institut
LMU München
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Mathematical Quantum Mechanics

Problem Sheet 3

Hand-in deadline: 08.11.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: (a) Suppose $A : \mathcal{H} \to \mathcal{H}$ is a bounded operator. Show that AA^* and A^*A are self-adjoint and that

$$\sigma(AA^*) \cup \{0\} = \sigma(A^*A) \cup \{0\}. \tag{1}$$

(b) Suppose that $A : \mathcal{H} \supset \mathcal{D}(A) \to \mathcal{H}$ is densely defined and define an operator $Q : \mathcal{H} \oplus \mathcal{H} \supset \mathcal{D}(Q) \to \mathcal{H} \oplus \mathcal{H}$ by

$$Q := \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \quad \mathcal{D}(Q) := \mathcal{D}(A) \oplus \mathcal{D}(A^*).$$

Prove that Q is self-adjoint if and only if A is closed.

(c) Suppose that $A : \mathcal{H} \supset \mathcal{D}(A) \to \mathcal{H}$ is densely defined and closed. Prove that AA^* and A^*A are self-adjoint. (*Hint:* You may use without proof that if Q is self-adjoint, then Q^2 is self-adjoint.[1])

Exercise 2: Suppose A is closed and B bounded.

(a) Show that if $\|B\| < 1$, then $I + B$ has a bounded inverse.

(b) Suppose that A has a bounded inverse. Show that if $\|B\| < \|A^{-1}\|^{-1}$, then $A + B$ has a bounded inverse.

Exercise 3: Let P be the momentum operator for a partcle on the real axis, i.e.

$$P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathcal{D}(B) = H^{1,2}(\mathbb{R}).$$

Let $f : \mathbb{R} \to \mathbb{C}$ be a Borel-measurable function and define $f(P) : \mathcal{D}_f \to L^2(\mathbb{R})$ by

$$f(P)\psi = \mathcal{F}^{-1}(f\widehat{\psi}), \quad \mathcal{D}_f = \{\psi \in L^2(\mathbb{R}) : f\widehat{\psi} \in L^2(\mathbb{R})\}.$$

[1] This follows from the spectral theorem.

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Prove the following:

(a) The map $f \mapsto f(P)$ is a $*$-homomorphism between the algebra of bounded measurable functions and the algebra of bounded linear operators on $\mathcal{H} = L^2(\mathbb{R})$ (denoted by $BL(\mathcal{H})$ in the following).

(b) The map

$$\begin{aligned} \mathcal{B}(\mathbb{R}) &\to BL(\mathcal{H}) \\ f &\mapsto \mathbf{1}_B(P) \end{aligned}$$

is a spectral measure. Here $\mathcal{B}(\mathbb{R})$ denotes the Borel sets.

(c) $\mathcal{D}_{\mathrm{id}} = H^{1,2}(\mathbb{R})$ and $\mathrm{id}(P) = P$.

(d) For $\psi \in L^2(\mathbb{R})$, $\|\psi\| = 1$,

$$\mu_\psi := \langle \psi, \mathbf{1}_\bullet(P)\psi \rangle$$

defines a probability measure on $\mathbb{R}$, and for $\psi \in \mathcal{D}_f$ it holds that

$$\|f(P)\psi\|^2 = \int_{\mathbb{R}} \mathrm{d}\mu_\psi(p) |f(p)|^2.$$