Mathematisches Institut
LMU München
WS 2017-18

Prof. P. Müller
Dr. R. Helling
Dr. J.-C. Cuenin

Mathematical Quantum Mechanics

Problem Sheet 2

Hand-in deadline: 02.11.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: (a) Let A be a closed, densely defined (linear[1]) operator. Prove that

$$\sigma(A^*) = \{\overline{z} : z \in \sigma(A)\}$$

and that for every $z \in \rho(A) = \mathbb{C} \setminus \sigma(A)$ one has

$$\big((A - z)^{-1}\big)^* = (A^* - \overline{z})^{-1}.$$

(b) Prove that if A is densely defined, then A^* is closed.

(c) Let A be symmetric. Prove that for all $\psi \in \mathcal{D}(A)$ and for all $z \in \mathbb{C} \setminus \mathbb{R}$, one has

$$\|(A - z)\psi\| \geq |\mathrm{Im} z|\, \|\psi\|.$$

Under the assumption that A is self-adjoint, deduce an upper bound for the norm of the resolvent operator $(A - z)^{-1}$.

Exercise 2: Let A be a symmetric positive definite real $d \times d$ matrix and let $b \in \mathbb{R}^d$. Define

$$f(x) := \mathrm{e}^{-x\cdot Ax + b\cdot x}, \quad x \in \mathbb{R}^d.$$

Show that the Fourier transform $\widehat{f}(k) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} f(x)\, \mathrm{d}x$ is given by

$$\widehat{f}(k) = \frac{1}{2^{d/2}\sqrt{\det A}} \mathrm{e}^{-\frac{1}{4}(k+\mathrm{i}b)\cdot(A^{-1}(k+\mathrm{i}b))}.$$

Exercise 3: Prove that $P_0^* = P$, where

$$P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathcal{D}(P) = H^{1,2}(\mathbb{R}),$$
$$P_0 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathcal{D}(P_0) = C_c^\infty(\mathbb{R}).$$

[1] All operators will be assumed to be linear.

Mathematical Quantum Mechanics

Exercise 4: Definition: Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. We say that $\lambda \in \mathbb{C}$ is an *approximate eigenvalue* of A if there exists a sequence $(\psi)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ such that $\|\psi_n\| = 1$ and $\|(A-\lambda)\psi_n\| \to 0$ as $n \to \infty$. We call such a sequence a *Weyl sequence* (or singular sequence) for A and λ.

(a) Prove that if $\lambda \in \mathbb{C}$ is an approximate eigenvalue of A, then $\lambda \in \sigma(A)$.

(b) The converse is also true (you don't need to prove this). As an example, let $A = -\Delta$ be the Laplacian on $\mathbb{R}^d$ with domain $H^{2,2}(\mathbb{R}^d)$. Construct a Weyl sequence for each $\lambda > 0$.