Mathematisches Institut — Prof. P. Müller
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Mathematical Quantum Mechanics

Problem Sheet 13

Hand-in deadline: 31.01.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: Suppose there is a physical device that is characterized by a non-linear map N on density operators, i.e., one for which there exists a convex decomposition of a density operator $\rho = \sum_{i=1}^{N} \lambda_i \rho_i$ such that

$$N\left(\sum_{i=1}^{N} \lambda_i \rho_i\right) \neq \sum_{i=1}^{N} \lambda_i N(\rho_i).$$

Recall the quantum steering result, and assume that ρ is Alice's reduced state of a bipartite pure state ψ. Alice applies her non-linear device and Bob does nothing, then $N(\rho)$ describes Alice's. If, however, Bob applies an instrument $\{T_i\}$ tailored to prepare ρ_i on Alice's side with probability λ_i, then Alice's state will be $\sum_{i=1}^{N} \lambda_i N(\rho_i)$. Argue why this implies that Alice can gain information about whether or not Bob applied the instrument, conflicting with the no-signaling condition.

Exercise 2: In the following, $\mathcal{H}$ is a finite dimensional Hilbert space. A uniformly continuous *quantum Markov semigroup* (QMS) of $\mathcal{A} := \mathcal{B}(\mathcal{H})$ is a norm continuous one-parameter semigroup $\mathbb{R}_+ \ni t \mapsto \mathcal{T}_t$ of completely positive (CP) linear maps such that $\mathcal{T}_t(1) = 1$ and $\mathcal{T}_0 = \mathrm{id}$. The purpose of this exercise is to prove the following theorem of Gorini-Kossakowski-Sudarshan and Lindblad.

Let $\mathcal{L}$ be the generator of the QMS, $\mathcal{T}_t = \mathrm{e}^{t\mathcal{L}}$.

Note the following which can be seen by identifying $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$ with $\mathcal{M}_n(\mathcal{A})$: Φ is a CP map iff any $n \in \mathbb{N}$, for any $X_1, \dots X_n \in \mathcal{A}$ and $\psi_1, \dots, \psi_n \in \mathcal{H}$,

$$\sum_{i,j} \langle \psi_i, \Phi(X_i^* X_j) \psi_j \rangle \geq 0.$$

Theorem. *If $\mathcal{T}_t$ is a QMS, then there exists a family $L_k \in \mathcal{A}$ and a hermitian $H = H^* \in \mathcal{A}$ such that*

$$\mathcal{L}(X) = \mathrm{i}[H, X] - \frac{1}{2} \sum_k \left(L_k^* L_k X - 2L_k^* X L_k + X L_k^* L_k\right)$$

for all $X \in \mathcal{B}(\mathcal{H})$. Reciprocally, any map $\mathcal{L}$ of this form generates a QMS.

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1. For any unital CP map Φ, prove that for any $n \in \mathbb{N}$, for any $X_1, \dots X_n \in \mathcal{A}$ and $\psi_1, \dots, \psi_n \in \mathcal{H}$,

$$\sum_{i,j} \langle \psi_i, (\Phi(X_i^* X_j) - \Phi(X_i)^* \Phi(X_j))\, \psi_j \rangle \geq 0.$$

   (Hint: use Stinespring's dilation)

2. Prove that the generator $\mathcal{L}$ of a QMS is conditionally completely positive (CCP), namely for any $n \in \mathbb{N}$, any $X_1, \dots X_n \in \mathcal{A}$ and $\psi_1, \dots, \psi_n \in \mathcal{H}$ such that $\sum_i X_i \psi_i = 0$,

$$\sum_{i,j} \langle \psi_i, \mathcal{L}(X_i^* X_j) \psi_j \rangle \geq 0.$$

   Moreover, show that

$$\mathcal{L}(1) = 0.$$

3. Prove that for any CCP map such that $\mathcal{L}(X^*) = \mathcal{L}(X)^*$, there is an operator $G \in \mathcal{A}$ and a CP map Φ such that

$$\mathcal{L}(X) = G^* X + \Phi(X) + XG$$

   (Hint: Define G^* as follows: Let $\psi_0 \in \mathcal{H}$, $\|\psi_0\| = 1$ and

$$G^* \psi := \mathcal{L}\left(|\psi\rangle\langle\psi_0|\right)\psi_0 - \frac{1}{2}\langle \psi_0, \mathcal{L}\left(|\psi_0\rangle\langle\psi_0|\right)\psi_0 \rangle\, \psi;$$

   Moreover, for any $X_1, \dots X_n \in \mathcal{A}$ and $\psi_1, \dots, \psi_n$, let $\psi_{n+1} := \psi_0$ and choose a suitable X_{n+1}; Check that Φ defined above is CP)

4. With this, prove the necessity part of the theorem

5. Prove the sufficiency part of the theorem. (Hint: $\mathcal{L} = \sum_{j \geq 0} \mathcal{L}_j$ where

$$\begin{aligned}
\mathcal{L}_0^{(1)}(X) &:= \mathrm{i}[H, X] \\
\mathcal{L}_0^{(2)}(X) &:= -\frac{1}{2} \sum_k \left(L_k^* L_k X + X L_k^* L_k\right) \\
\mathcal{L}_j(X) &:= L_j^* X L_j, \qquad j \geq 1.
\end{aligned}$$

   Use Trotter's product formula.)