Mathematisches Institut
LMU München
WS 2017-18

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Mathematical Quantum Mechanics

Problem Sheet 11

Hand-in deadline: 17.01.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: (This exercise counts for 8 points.) Recall the following
Definition: We say that a formal power series $\sum_n a_n z^n$ is *Borel summable* if:

1. $B(t) := \sum_n a_n t^n/n!$ converges in some circle $|t| < \delta$;
2. $B(t)$ has an analytic continuation to a neighborhood of the positive real axis;
3. $g(z) := (1/z) \int_0^x \mathrm{e}^{-t/z} B(t)\, \mathrm{d}t$ converges (not necessarily absolutely) for some $z \neq 0$.

Then, $B(t)$ is called the *Borel transform* of the series $\sum_n a_n z^n$ and $g(z)$ is called its *Borel sum*.

The aim of this exercise sheet is to prove the following

Theorem: Let $C_R := \{z \in \mathbb{C} : \mathrm{Re}(1/z) > 1/R\}$ for $R > 0$. Assume that $f : C_R \to \mathbb{C}$ is analytic and has the asymptotic expansion

$$f(z) = \sum_{k=0}^{N-1} a_k z^k + R_N(z),$$

with

$$|R_N(z)| \leq A\sigma^N N! |z|^N,$$

uniformly in $N \in \mathbb{N}$ and in $z \in C_R$, for some constants $A, \sigma > 0$. Then the asymptotic expansion of f is Borel summable. More precisely, $B(t) = \sum_n a_n t^n/n!$ converges for $|t| < \sigma$ and has an analytic continuation to the region

$$S_\sigma := \{t \in \mathbb{C} : \mathrm{dist}(t, \mathbb{R}_+) < 1/\sigma\}, \tag{1}$$

satisfying the bound

$$|B(t)| \leq K \exp(t/R) \tag{2}$$

uniformly in every $S_{\sigma'}$ with $\sigma' > \sigma$.[1] Furthermore, f can be represented by the absolutely convergent integral

$$f(z) = (1/z)\int_0^\infty \mathrm{e}^{-t/z} B(t)\,\mathrm{d}t, \quad z \in C_R. \tag{3}$$

(a) Show that C_R is a circle tangent to the imaginary axis, and sketch the region S_σ in (1).

(b) Consider

$$b_m(t) := a_m + \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}(1/z)=1/r} \mathrm{e}^{t/z} z^{-(m+1)} \left(f(z) - \sum_{k=0}^{m} a_k z^k \right) \mathrm{d}z$$

where the integral is understood as a contour integral. Prove that the integral is absolutely convergent for $t \geq 0$ and independent of r for $0 < r < R$.

(c) Prove that b_0 is a smooth function with m-th derivative equal to b_m and that

$$|b_m(t)| \leq K_1 \sigma^{m+1}(m+1)!\exp(t/R),$$

with K_1 independent of t and m.

(d) Prove that

$$b_0(t) = \sum_{k=0}^{N-1} a_k t^k/k! + \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}(1/z)=1/r} \mathrm{e}^{t/z} z^{-1} R_N(z)\,\mathrm{d}z$$

and that the remainder satisfies

$$\left| b_0(t) - \sum_{k=0}^{N-1} a_k t^k/k! \right| \leq K_2\sqrt{N}(\sigma t)^N \to 0 \quad \text{as } N \to \infty \quad \text{for } 0 \leq t < 1/\sigma.$$

Hint: Take $r = t/N$ and use Stirling's formula.

(e) Conclude that $B(t) = b_0(t)$ for $0 \leq t < 1/\sigma$. Moreover, for $t_0 \geq 0$ define

$$B_{t_0}(t) := \sum_{m=0}^{\infty} b_m(t_0)\frac{(t-t_0)^m}{m!}.$$

Prove that the series is absolutely convergent for $|t - t_0| < 1/\sigma$ and that $B_{t_0}(t) = B_{t_1}(t)$ whenever both are defined. Use this to construct an analytic continuation of $B(t)$ to S_σ which satisfies the bound (2).

(f) Prove (3). *Hint:* Use $B(t) = b_0(t)$ and Cauchy's formula.

(g) Consider $f(z) = \exp(-z^{-a})$, with $a > 0$, and check whether this function satisfies the assumptions of the theorem.

[1] i.e. for every $\sigma' > \sigma$ there exists $K = K(\sigma')$ such that (2) holds.