

Mathematical Quantum Mechanics

Problem Sheet 10

Hand-in deadline: 10.01.2017 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: Let D be the closure of the symmetric operator $D_0 : C_c^\infty(\mathbb{R}^d \setminus \{0\}) \rightarrow L^2(\mathbb{R}^d)$ defined by

$$(D_0\psi)(x) := \frac{-i}{2} \sum_{\nu=1}^d \left(x_\nu \frac{\partial}{\partial x_\nu} + \frac{\partial}{\partial x_\nu} x_\nu \right) \psi(x), \quad \forall x \in \mathbb{R}^d.$$

Then

1. $D = D^*$;
2. $\mathcal{F}D\mathcal{F}^* = -D$;
3. For $\tau \in \mathbb{R}$ let $\mathcal{U}(\tau) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ be given by

$$(\mathcal{U}(\tau)\psi)(x) := e^{d\tau/2} \psi(e^\tau x), \quad \forall x \in \mathbb{R}^d.$$

Then $\{\mathcal{U}(\tau)\}_{\tau \in \mathbb{R}}$ is a strongly continuous unitary group with generator D , i.e. $\mathcal{U}(\tau) = e^{iD\tau}$ for all $\tau \in \mathbb{R}$.

Exercise 2: Define the Mellin transform

$$\mathcal{M} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R} \times \mathcal{S}^{d-1})$$

by

$$(\mathcal{M}\psi)(\lambda, \omega) := \frac{1}{\sqrt{2\pi}} \text{l.i.m.}_{N \rightarrow \infty} \int_{1/N}^N r^{-i\lambda-1+d/2} \psi(r\omega) dr, \quad \forall (\lambda, \omega) \in \mathbb{R} \times \mathcal{S}^{d-1}.$$

Then

1. \mathcal{M} is unitary with inverse

$$(\mathcal{M}^{-1}\varphi)(r\omega) := \frac{r^{-d/2}}{\sqrt{2\pi}} \text{l.i.m.}_{N \rightarrow \infty} \int_{-N}^N r^{-i\lambda} \varphi(\lambda, \omega) d\lambda, \quad \forall (r, \omega) \in \mathbb{R}_{>0} \times \mathcal{S}^{d-1}.$$

2. For every $\tau \in \mathbb{R}$ and $\varphi \in L^2(\mathbb{R} \times \mathcal{S}^{d-1})$ we have

$$(\mathcal{M}\mathcal{U}(\tau)\mathcal{M}^{-1}\varphi)(\lambda, \omega) = e^{i\tau\lambda} \varphi(\lambda, \omega), \quad \forall (\lambda, \omega) \in \mathbb{R} \times \mathcal{S}^{d-1}$$

and

$$(\mathcal{M}D\mathcal{M}^{-1}\varphi)(\lambda, \omega) = \lambda\varphi(\lambda, \omega), \quad \forall (\lambda, \omega) \in \mathbb{R} \times \mathcal{S}^{d-1}.$$