MATHEMATISCHES INSTITUT
LMU MÜNCHEN
WS 2016-17

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Mathematical Quantum Mechanics

Problem Sheet 7

Hand-in deadline: 12/08/2016 before noon in the designated MQM box (1st floor, next to the library).

For this sheet you may use the following fact.

Min-max principles: Consider the Schrödinger operator $H = -\Delta + V$ defined on the dense domain $\mathcal{D}(H) = C_c^2(\mathbb{R}^d)$, and let $\mathcal{E}(\psi) = \langle \psi, H\psi \rangle$ for $\psi \in \mathcal{D}(H)$. Here we assume that $V \in L^2_{\text{loc}}(\mathbb{R}^d)$ and that the negative part $V_- = \min\{V, 0\}$ satisfies $V_- \in L^{d/2+1}(\mathbb{R}^d)$. Define the eigenvalues of H in the following way: $E_0 = \inf\{\mathcal{E}(\psi) : \psi \in \mathcal{D}(H),\ \|\psi\| = 1\}$. If the infimum is attained for some $\psi_0 \in \mathcal{D}(H)$, we go on to define $E_1 = \inf\{\mathcal{E}(\psi) : \psi \in \mathcal{D}(H),\ \|\psi\| = 1,\ \langle \psi, \psi_0 \rangle = 0\}$ and so on. Let J be the first integer for which the infimum is not attained. Then we define $E_k = E_J$ for $k \geq J$.[1] Define the *max-min* values λ_N and the *min-max* values μ_N by

$$\begin{aligned}
\lambda_N &= \max_{\phi_0, \ldots, \phi_{N-1}} \min\{\mathcal{E}(\phi) : \|\phi\| = 1,\ \langle \phi, \phi_k \rangle = 0 \text{ for all } k = 0, \ldots, \phi_{N-1}\}, \\
\mu_N &= \min_{\phi_0, \ldots, \phi_N} \max\{\mathcal{E}(\phi) : \|\phi\| = 1,\ \phi \in \operatorname{span}(\phi_0, \ldots, \phi_N)\},
\end{aligned}$$

where the maximum in the first and the minimum in the second expression are taken over all orthonormal functions $\phi_0, \ldots, \phi_N \in C_c^2(\mathbb{R}^d)$. Then the following hold.

(i) If $N < J$, then $\lambda_N = E_N$.

(ii) If $N < J$, then $\mu_N = E_N$.

Exercise 1: *Lieb-Thirring inequality:* Let H be as above. The aim of this exercise is to prove the following inequality on the sum of negative eigenvalues of H, known as the Lieb-Thirring inequality,

$$\sum_{j=0}^{\infty} (E_j)_- \geq -C_d \int_{\mathbb{R}^d} |V_-(x)|^{d/2+1} dx. \tag{1}$$

[1] Incidentally, the number E_J is called the *bottom of the essential spectrum of H*.

Proceed in the following way:

(i) Assume that

$$\sum_{j=0}^{N} \langle \phi_j, H\phi_j \rangle \geq -C_d \int_{\mathbb{R}^d} |V_-(x)|^{d/2+1} dx \tag{2}$$

holds for all finite orthonormal families $\{\phi_j\}_{j=0}^N$ in $C_c^2(\mathbb{R}^d)$. Prove that (1) follows.

(ii) For $\phi \in L^2(\mathbb{R}^d)$ and $e > 0$, define $\widehat{\phi^{e,+}}(\xi) = \mathbf{1}\{\xi : |\xi|^2 \geq e\}\widehat{\phi}(\xi)$ and $\widehat{\phi^{e,-}} = \widehat{\phi} - \widehat{\phi^{e,+}}$. Prove that

$$\int_0^\infty \int_{\mathbb{R}^d} |\phi^{e,+}(x)|^2 dx de = \int_{\mathbb{R}^d} |\nabla \phi|^2 dx.$$

(iii) Prove that for any orthonormal family $\{\phi_j\}_{j=0}^N$ in $L^2(\mathbb{R}^d)$ and for $x \in \mathbb{R}^d$,

$$\left(\sum_{j=0}^{N} |\phi_j^{e,+}(x)|^2 \right)^{1/2} \geq \left[\left(\sum_{j=0}^{N} |\phi_j(x)|^2 \right)^{1/2} - \left(\sum_{j=0}^{N} |\phi_j^{e,-}(x)|^2 \right)^{1/2} \right]_+ .$$

(iv) Prove that for any orthonormal family $\{\phi_j\}_{j=0}^N$ in $C_c^2(\mathbb{R}^d)$ and for $x \in \mathbb{R}^d$,

$$\sum_{j=0}^{N} |\phi_j^{e,-}(x)|^2 \leq (2\pi)^{-d} \kappa_d e^{d/2},$$

where κ_d is the surface measure on S^{d-1}.

(v) Conclude from (ii)-(iv) that

$$\sum_{j=0}^{N} \int_{\mathbb{R}^d} |\nabla \phi_j(x)|^2 dx \geq \frac{(2\pi)^2 d^2 \kappa_d^{-2/d}}{(d+2)(d+4)} \int_{\mathbb{R}^d} \left(\sum_{j=0}^{N} |\phi_j(x)|^2 \right)^{(d+2)/d} dx.$$

(vi) Prove (2).

Exercise 2: Let $(f_j)_{j=0}^N$ be orthonormal functions such that $f_i \in H^1(\mathbb{R}^d)$ for $1 \leq i \leq N$. Let $\psi = f_1 \wedge \cdots \wedge f_N$, and $\rho_\psi(x) = \gamma_\psi^{(1)}(x,x)$. Show that there exists $K_d > 0$ such that

$$T(\psi) \geq K_d \int_{\mathbb{R}^d} \rho_\psi(x)^{1+\frac{2}{d}} dx$$

where $T(\psi) = \sum_{i=1}^N \int |\nabla_{x_i}\psi|^2$ (*Hint:* Consider the 'potential' $U(x) = c\rho_\psi(x)^{\frac{2}{d}}$ and use the Lieb-Thirring inequality).