Mathematisches Institut | Prof. H. Siedentop
LMU München | Dr. R. Helling
WS 2016-17 | Dr. J.-C. Cuenin

Mathematical Quantum Mechanics

Problem Sheet 4

Hand-in deadline: 11/17/2016 before noon in the designated MQM box (1st floor, next to the library).

Exercise 1: *Banach-Alaoglu theorem for separable Hilbert spaces.* Let $\mathfrak{H}$ be a separable Hilbert space. Prove that every bounded sequence in $\mathfrak{H}$ has a weakly convergent subsequence. Proceed as follows:

(a) Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathfrak{H}$. Prove that for any bounded sequence $\{h^n\}_{n\in\mathbb{N}}$ in $\mathfrak{H}$ there is a subsequence $\{h^{n_m}\}_{m\in\mathbb{N}}$ such that the limit $\lim_{m\to\infty}\langle e_j, h^{n_m}\rangle$ exists for each $j\in\mathbb{N}$. *Hint:* Cantor diagonal process.

(b) Denote $k^m := h^{n_m}$ and $a_j := \lim_{m\to\infty}\langle e_j, k^m\rangle$. Prove that $\sum_{j=1}^\infty |a_j|^2$ is convergent. *Hint:* Prove that

$$\sum_{j=1}^N |a_j|^2 = \lim_{m\to\infty}\left\langle k^m, \sum_{j=1}^N a_j e_j\right\rangle$$

and estimate the limit by the Schwarz inequality.

(c) Prove that $h := \sum_{j=1}^\infty a_j e_j$ exists as an element in $\mathfrak{H}$ and that k^m converges weakly to h as $m\to\infty$.

Exercise 2: *The Foldy-Wouthuysen transformation.* Let D_0 be the free Dirac operator on $L^2(\mathbb{R}^3;\mathbb{C}^4)$ with domain $\mathcal{D}(D_0) = H^1(\mathbb{R}^3;\mathbb{C}^4)$ (see the lecture). Find a unitary operator U_{FW} on $L^2(\mathbb{R}^3;\mathbb{C}^4)$ that diagonalizes D_0, v.i.z.

$$U_{FW} D_0 U_{FW}^* = \begin{pmatrix} \sqrt{-\Delta+m^2}\otimes \mathbf{1}_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & -\sqrt{-\Delta+m^2}\otimes \mathbf{1}_{2\times2}\end{pmatrix}.$$

Exercise 3: *The Schrödinger operator for Helium.* Let $Z>0$. Prove that the Schrödinger operator

$$H_Z := -\frac{1}{2}\Delta_x - \frac{1}{2}\Delta_y - \frac{Z}{|x|} - \frac{Z}{|y|} + \frac{1}{|x-y|}$$

is selfadjoint on the domain $\mathcal{D}(H_Z) := H^2(\mathbb{R}^3\times\mathbb{R}^3)$. Here, H_2 is the Hamiltonian for a Helium atom. Can this be generalized to other atoms or molecules?