MATHEMATISCHES INSTITUT
LMU MÜNCHEN
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Mathematical Quantum Mechanics

Problem Sheet 2

Hand-in deadline: 11/03/2016 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1:

1. Let $\Omega = \mathbb{R}^d \setminus \{0\}$. Let $V \in L^1_{\text{loc}}(\Omega)$ and assume that there is a positive radial function $\omega \in C^2(\Omega)$ satisfying $-\Delta\omega + V\omega \geq 0$ in Ω. In addition, assume the following (technical) conditions:

   - $\lim_{\epsilon\to 0} \frac{\omega'(\epsilon)}{\omega(\epsilon)}\epsilon^{d-1} = 0$.
   - $r \mapsto \left(\frac{\omega'(r)}{\omega(r)}\right)^2 r^{d-1}$ is integrable at the origin.

   Prove that
   $$\int_{\mathbb{R}^d} (|\nabla\phi|^2 + V|\phi|^2)\mathrm{d}x \geq \int_{\mathbb{R}^d} |\nabla(\omega^{-1}\phi)|^2\omega^2 \mathrm{d}x \qquad \text{for all } \phi \in C^1_c(\mathbb{R}^d).$$

2. Let $d \geqslant 3$. Use the first part of the exercise to prove that
   $$\int_{\mathbb{R}^d} \left|\nabla\phi(x)\right|^2 \mathrm{d}x \geq \left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{\left|\phi(x)\right|^2}{|x|^2}\mathrm{d}x \qquad \text{for all } \phi \in C^1_c(\mathbb{R}^d).$$
   This is *Hardy's inequality (quantum mechanical uncertainty principle)*.

3. Let $d \geq 2$. Use the first part of the exercise to prove that
   $$\frac{d-1}{2}\int_{\mathbb{R}^d} \frac{\left|\phi(x)\right|^2}{|x|}\mathrm{d}x \leq \left(\int_{\mathbb{R}^d} |\nabla\phi|^2 \mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}^d} |\phi|^2 \mathrm{d}x\right)^{1/2}$$
   for all $\phi \in C^1_c(\mathbb{R}^d)$. *Hint: Choose* $\omega(x) = \exp(-\alpha|x|)$ *for some* $\alpha > 0$.

Exercise 2:

1. Prove that
   $$c\int_{\mathbb{R}^3} \frac{1}{|x|}\psi(x)\mathrm{d}x = \int_{\mathbb{R}^3} \frac{1}{|k|^2}\widehat{\psi}(k)\mathrm{d}k,$$
   $$\int_{\mathbb{R}^3} \frac{1}{|x|^2}\psi(x)\mathrm{d}x = c\int_{\mathbb{R}^3} \frac{1}{|k|}\widehat{\psi}(k)\mathrm{d}k$$
   for some constant $c > 0$ and for all $\psi \in \mathcal{S}(\mathbb{R}^3)$. Compute c.

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2. Prove that

$$\int_{\mathbb{R}^3} \frac{|\psi|^2}{|x|} \mathrm{d}x \leq C\langle\psi, |\mathbf{p}|\psi\rangle \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}^d)$$

for some constant $C > 0$. This inequality with the sharp constant is sometimes called *Kato's inequality*. Here, the operator $|\mathbf{p}|$ is defined as

$$\mathcal{F}\left(|\mathbf{p}|\psi\right)(k) := |k|\widehat{\psi}(k)$$

for $\psi \in \mathcal{S}(\mathbb{R}^3)$. (Remark: The sharp constant is $C = \pi/2$.)

Jean-Claude Cuenin