Mathematisches Institut
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Mathematical Quantum Mechanics

Problem Sheet 10

Hand-in deadline: 01/12/2017 before noon in the designated MQM box (1st floor, next to the library).

Reminder: Recall the following fact from the tutorial. Let $E_0^f(N)$ be the ground state energy of N non-interacting fermions, where the one-particle Hamiltonian is given by $H = -\Delta + V$, with $V \in L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ decaying at infinity, and where H has at least N negative eigenvalues $-\infty < E_0 \leq E_1 \leq E_2 \leq \ldots$ Then

$$E_0^f(N) = \sum_{j=0}^{N-1} E_j = \inf\{\mathrm{tr}[H\gamma] : 0 \leq \gamma \leq \mathbf{1},\, \mathrm{tr}[\gamma] = N,\, \gamma = \sum_{j\geq 1} \mu_j |\psi_j\rangle\langle\psi_j|$$
$$\text{(spectral decomposition) with } \psi_j \in H^1(\mathbb{R}^d),\, \sum_{j\geq 1} \mu_j \|\nabla\psi_j\|^2 < \infty\}.$$

Ex. 1: Let $g \in \mathcal{S}(\mathbb{R}^d)$ satisfy the following properties

- $g(x) \geq 0$, $g(x) = g(-x)$,
- $g \in L^2(\mathbb{R}^d)$, $\|g\|_2 = 1$.

For any $\theta > 0$ define $g_\theta(x) := \theta^{-d/2} g(x/\theta)$ for $x \in \mathbb{R}^d$. For $(q,k) \in \mathbb{R}^d \times \mathbb{R}^d$ we define the function $\psi_{q,k}(x) := g(x-q)\exp(2\pi i k \cdot x)$ for $x \in \mathbb{R}^d$. Given a bounded measurable function $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we define the *coherent state quantization*[1] of a to be the bounded operator $Op(a)$ given by

$$Op(a)\psi := \int\int a(q,k)|\psi_{q,k}\rangle\langle\psi_{q,k}|\psi\rangle \mathrm{d}q\mathrm{d}k, \quad \psi \in L^2(\mathbb{R}^d).$$

(a) Show that $Op(1) = \mathbf{1}$.

(b) Show that if $c \leq a(q,k) \leq C$, then $c\mathbf{1} \leq Op(a) \leq C\mathbf{1}$.

(c) Show that $\mathrm{tr}[Op(a)] = \int\int a(q,k)\mathrm{d}q\mathrm{d}k$ for $a \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$.

(d) Show that $\mathrm{tr}[(-\Delta)Op(a)] = \int\int a(q,k)\|\nabla\psi_{q,k}\|^2\mathrm{d}q\mathrm{d}k$

[1] sometimes also called *anti-Wick quantization.*

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Ex. 2: Let $d \geq 1$. Let V be a bounded potential such that $V_- \in L^{\frac{d+2}{2}}(\mathbb{R}^d)$.[2] Assume that $H_h := -h^2\Delta + V$ has only discrete spectrum (isolated eigenvalues of finite multiplicity) below 0 that may accumulate only at 0 (This condition is fulfilled e.g if $\liminf_{|x|\to\infty} V \geq 0$.) Here, $h > 0$ is a (semi-classical) parameter which we will let tend to zero. We denote the negative eigenvalues of H_h by $-\infty < E_0(h) \leq E_1(h) \leq E_2(h) \leq \ldots$ (counting multiplicities). The *Weyl law* for the sum of negative eigenvalues of H_h sates that

$$\sum_{j=0}^{\infty}(E_j)_- = \int\int [p^2 + V(x)]_- \frac{\mathrm{d}x\mathrm{d}p}{(2\pi h)^d} + o(h^{-d}).$$

The aim of this exercise is to prove this statement.

(a) For the upper bound, use the trial density matrix $\gamma = Op(\mathbf{1}_{h^2p^2+V(x)\leq 0})$.

(b) For the lower bound, split H_h as $H_h = H^{(0)} + H^{(1)}$ where

$$H^{(0)} = -(1-\delta)h^2\Delta + V_R * g_\theta^2 + (1-\delta)h^2 W_{\theta,R}(x),$$
$$H^{(1)} = -\delta h^2\Delta + V - V_R * g_\theta^2 - (1-\delta)h^2 W_{\theta,R}(x),$$

and where $V_R(x) = \mathbf{1}\{|x| \leq R\}V(x)$ and

$$W_{\theta,R}(x) = \int_{|u|<R} (\nabla g_\theta)^2(x-u) - \frac{1}{2}\Delta(g_\theta^2)(x-u)\mathrm{d}u.$$

Hint: Use the Lieb-Thirring inequality to estimate $H^{(1)}$.

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[2] Recall that $\frac{d+2}{2}$ is the exponent in the Lieb-Thirring inequality.