Mathematisches Institut
der Universität München

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WiSe 2015/16 **Mathematical Quantum Mechanics** 08.12.2015

Problem Sheet 9

Hand-in deadline: 15.12.2015 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: [The bathtub principle] Let $G > 0$ be fixed and let

$$\mathcal{C} := \left\{ g : 0 \leq g(x) \leq 1 \text{ for all } x \text{ and } \int_{\mathbb{R}^d} g(x)dx = G \right\}.$$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $|\{x : f(x) < t\}| < \infty$ for all $t \in \mathbb{R}$. Define

$$I := \inf_{g \in \mathcal{C}} \int_{\mathbb{R}^d} f(x)g(x)dx.$$

Prove that

$$I = \int_{f<S} f(x)dx + cS|\{x : f(x) = S\}|$$

where

$$\begin{aligned} S &= \sup \{t : |\{x : f(x) < t\}| \leq G\}, \\ c &= |\{x : f(x) = S\}|^{-1}\big(G - |\{x : f(x) < S\}|\big), \end{aligned}$$

and the minimiser is

$$g_0(x) := \chi_{\{f<S\}}(x) + c\chi_{\{f=S\}}(x).$$

Ex. 2: Let $\Omega \subset \mathbb{R}^d$ be open with $|\Omega| < \infty$ and let

$$E(\psi) := \int_\Omega |\nabla\psi(x)|^2 dx.$$

Let $\psi_0, \dots, \psi_{N-1}$ be a collection of $L^2(\Omega)$-orthonormal functions in $C_0^\infty(\Omega)$. Show that

$$\sum_{j=0}^{N-1} E(\psi_j) \geq (2\pi)^2 \frac{d}{d+2} \left(\frac{d}{|\mathbb{S}^{d-1}|} \right)^{2/d} N^{1+2/d} \frac{1}{|\Omega|^{2/d}}.$$

(Hint: Use the Fourier transform and the bathtub principle)

Jean-Claude Cuenin