MATHEMATISCHES INSTITUT
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Problem Sheet 8

Hand-in deadline: 01.12.2015 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: Let

$$G(x,y) := \frac{1}{4\pi}\frac{1}{|x-y|},$$

and for $f \in C_c^\infty(\mathbb{R}^3)$, let

$$\Phi_f(x) := (G * f)(x) = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{f(y)}{|x-y|}.$$

(i) Show that $\Delta_x G(x,y) = 0$ if $x \neq y$.

(ii) Prove that $\Phi_f \in C^\infty(\mathbb{R}^3)$.

(iii) Prove that $f = \Phi_{-\Delta f}$ and conclude that Φ_f is a solution of

$$-\Delta u(x) = f(x), \qquad x \in \mathbb{R}^3.$$

(iv) Prove that

$$\lim_{|x|\to\infty} \Phi_f(x)\Big(\frac{\int_{\mathbb{R}^3} f(y)dy}{4\pi|x|}\Big)^{-1} = 1$$

(v) Reciprocally, let $u \in H^2(\mathbb{R}^3)$ be a solution of $-\Delta u(x) = f(x)$. Prove that $u(x) = \Phi_f(x)$ almost everywhere (Hint: Consider $\langle \varphi, u\rangle$ for an arbitrary $\varphi \in C_c^\infty(\mathbb{R}^3)$ and use the Fourier transform with Ex. 4, Sheet 5)

Ex. 2: Recall the definition of the Sobolev space $H^\alpha(\mathbb{R}^d)$ (where d is a positive integer and $\alpha > 0$):

$$H^\alpha(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{H^\alpha}^2 := \int_{\mathbb{R}^d} (1+|k|^2)^\alpha |\widehat{f}(k)|^2 \,\mathrm{d}k < \infty \right\}.$$

Assume that $\alpha > \frac{d}{2} + l$ for some non-negative integer l. Prove that $H^\alpha(\mathbb{R}^d) \subset C^l(\mathbb{R}^d)$.

Ex. 3: Let $f_0, f_1, \ldots$, with $f_i \in H^1(\mathbb{R}^3), \langle f_i, f_j\rangle_{L^2(\mathbb{R}^3)} = \delta_{ij}$, be the eigenfunctions of $-\Delta + V$ for the negative eigenvalues $-e_0 \leq -e_1 \leq \ldots$. This is understood in the weak sense: For any $\varphi \in C_c^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} [(-\Delta + V(x))\varphi(x)] f_i(x) dx = -e_i \int_{\mathbb{R}^3} \varphi(x) f_i(x) dx.$$

Prove that

$$\int_{\mathbb{R}^3} \left[\overline{\nabla f_i(x)} \nabla f_j(x) + V(x) \overline{f_i(x)} f_j(x) \right] dx = -e_i \delta_{ij}.$$

(Hint: Use the density of $C_c^\infty(\mathbb{R}^3)$ in $H^1(\mathbb{R}^3)$)

Ex. 4: Consider the energy functional $E(\psi)$,

$$E(\psi) := \int_{\mathbb{R}} \left(|\psi'(x)|^2 + 9x^4(x^2+2)|\psi(x)|^2 \right) dx, \qquad \psi \in \mathcal{S}(\mathbb{R}),$$

and let $E_0 := \inf_{\psi \in \mathcal{S}(\mathbb{R}), \|\psi\|_2 = 1} E(\psi)$.

(i) Prove that $E(\psi) \geq 3\|\psi\|_2^2$ for all $\psi \in \mathcal{S}(\mathbb{R})$.

(ii) Prove that $E_0 = 3$ and that there exists a unique minimizer (up to a global phase) ψ_0, i.e. $E_0 = E(\psi_0)$.

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