MATHEMATISCHES INSTITUT
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Problem Sheet 5

Hand-in deadline: 17.11.2015 before 12:00 in the designated MQM box (1st floor, next to the library).

Ex. 1: Let A be a symmetric positive definite real $d \times d$ matrix and let $b \in \mathbb{R}^d$. Define

$$f(x) := \mathrm{e}^{-x\cdot Ax + b\cdot x}, \quad x \in \mathbb{R}^d.$$

Show that the Fourier transform $\widehat{f}(k) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} f(x)\, \mathrm{d}x$ is given by

$$\widehat{f}(k) = \frac{1}{2^{d/2}\sqrt{\det A}} \mathrm{e}^{-\frac{1}{4}(k+\mathrm{i}b)\cdot(A^{-1}(k+\mathrm{i}b))}.$$

Ex. 2: Recall that for any $\psi \in L^2(\mathbb{R}^d)$ and $t > 0$,

$$(\mathrm{e}^{\mathrm{i}t\Delta}\psi)(x) = \lim_{\epsilon\to 0} \frac{1}{(4\pi(\mathrm{i}t+\epsilon))^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4(\mathrm{i}t+\epsilon)}} \psi(y) dy,$$

where the limit is taken in $L^2(\mathbb{R}^d)$.

Assume now that $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

(i) Prove that

$$(\mathrm{e}^{\mathrm{i}t\Delta})\psi(x) = \frac{1}{(4\pi\mathrm{i}t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\frac{|x-y|^2}{4t}} \psi(y) dy$$

almost everywhere in $\mathbb{R}^d$.

(ii) Show that

$$\left\| \mathrm{e}^{\mathrm{i}t\Delta}\psi - \frac{\mathrm{e}^{\mathrm{i}|\cdot|^2/4t}}{(2\mathrm{i}t)^{d/2}} \widehat{\psi}\left(\frac{\cdot}{2t}\right) \right\|_{L^2(\mathbb{R}^d)} \xrightarrow{|t|\to\infty} 0$$

(iii) Assuming further that $|\cdot|^2\psi \in L^2(\mathbb{R}^d)$, show that there exits $C > 0$ such that

$$\left\| \mathrm{e}^{\mathrm{i}t\Delta}\psi - \frac{\mathrm{e}^{\mathrm{i}|\cdot|^2/4t}}{(2\mathrm{i}t)^{d/2}} \widehat{\psi}\left(\frac{\cdot}{2t}\right) \right\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{|t|}.$$

(iv) Assuming that $\psi \in \mathcal{S}(\mathbb{R}^d)$, show by an explicit computation using the right hand side above[1] that $\varphi(t,x) := (\mathrm{e}^{\mathrm{i}t\Delta}\psi)(x)$ is a solution of the partial differential equation

$$\mathrm{i}\partial_t\varphi(t,x) = -\Delta_x\varphi(t,x)$$

in $C^\infty((\mathbb{R}_t \setminus \{0\}) \times \mathbb{R}^d_x)$.

Ex. 3: (i) Consider the evolution $\mathrm{e}^{\mathrm{i}t\Delta}$ under the free propagator (see Ex. 2). Prove that for $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$,

$$\|\mathrm{e}^{\mathrm{i}t\Delta}\psi\|_2 = \|\psi\|_2 \quad \text{and} \quad \|\mathrm{e}^{\mathrm{i}t\Delta}\psi\|_\infty \leq \frac{1}{(4\pi t)^{d/2}}\|\psi\|_1 \quad \text{for all } t > 0.$$

(ii) Deduce from (i) that $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ implies that $\mathrm{e}^{\mathrm{i}t\Delta}\psi \in L^q(\mathbb{R}^d)$ for all $t > 0$ and $q \in [2,\infty]$, with

$$\|\mathrm{e}^{\mathrm{i}t\Delta}\psi\|_q \leq \frac{1}{(4\pi t)^{d(\frac{1}{2}-\frac{1}{q})}}\|\psi\|_1^{1-2/q}\|\psi\|_2^{2/q}.$$

Hint: Use Hölder's inequality to prove that if $\psi \in L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d)$ for some p_0, p_1 with $1 \leq p_0 < p_1 \leq \infty$, then $\psi \in L^p(\mathbb{R}^d)$ for all $p \in [p_0, p_1]$ and

$$\|\psi\|_p \leq \|\psi\|_{p_0}^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}}\|\psi\|_{p_1}^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}.$$

(iii) Use the Riesz-Thorin interpolation theorem (see last page) and the estimates in (i) to prove that for every $t > 0$, the propagator $\mathrm{e}^{\mathrm{i}t\Delta}$ extends uniquely to a bounded linear operator $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$, $p \in [1,2]$, $1/p + 1/q = 1$, with

$$\|\mathrm{e}^{\mathrm{i}t\Delta}\psi\|_q \leq \frac{1}{(4\pi t)^{d(\frac{1}{2}-\frac{1}{q})}}\|\psi\|_p.$$

Ex. 4: Prove that

$$c_3\int_{\mathbb{R}^3}\frac{1}{|x|}\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^3}\frac{1}{|k|^2}\widehat{\psi}(k)\,\mathrm{d}k,$$
$$\int_{\mathbb{R}^3}\frac{1}{|x|^2}\psi(x)\,\mathrm{d}x = c_3\int_{\mathbb{R}^3}\frac{1}{|k|}\widehat{\psi}(k)\,\mathrm{d}k$$

for some constant $c_3 > 0$ and for all $\psi \in \mathcal{S}(\mathbb{R}^3)$. Compute c_3.

[1] Do not use Stone's theorem; prove this here by means of elementary analysis.

THE RIESZ-THORIN THEOREM. Let $p_0, p_1, q_0, q_1 \in [1, \infty]$ and let $\theta \in (0, 1)$. Define $p, q \in [1, \infty]$ by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Assume that T is a linear operator[2] with

$$T : L^{p_0} \to L^{q_0}, \quad \|T\|_{L^{p_0} \to L^{q_0}} = M_0,$$
$$T : L^{p_1} \to L^{q_1}, \quad \|T\|_{L^{p_1} \to L^{q_1}} = M_1.$$

Then T extends uniquely to a bounded operator from L^p to L^q, and

$$\|T\|_{L^p \to L^q} \leq M_0^{1-\theta} M_1^{\theta}.$$

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[2]We fix a pair of measure spaces (X, μ) and (Y, ν). These measure spaces are suppressed in our notation. More precisely, $L^{p_i} \equiv L^{p_i}(X, \mu)$ and $L^{q_i} \equiv L^{q_i}(Y, \nu)$, $i = 0, 1$.